



ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Linear Algebra and its Applications 423 (2007) 456–497

LINEAR ALGEBRA
AND ITS
APPLICATIONSwww.elsevier.com/locate/laa

Ascent, descent, nullity, defect, and related notions for linear relations in linear spaces

Adrian Sandovici ^a, Henk de Snoo ^{b,*}, Henrik Winkler ^{c,1}^a *Department of Mathematics, University of Bacău, Str. Spiru Haret, nr. 8, 600114 Bacău, România*^b *Department of Mathematics and Computing Science, University of Groningen, P.O. Box 800,
9700 AV Groningen, The Netherlands*^c *Institut für Mathematik, MA 6-4, Technische Universität Berlin, Strasse des 17. Juni 136,
D-10623 Berlin, Deutschland, Germany*

Received 16 November 2005; accepted 31 January 2007

Available online 20 February 2007

Submitted by L. Rodman

Abstract

For a linear relation in a linear space the concepts of ascent, descent, nullity, and defect are introduced and studied. It is shown that the results of A.E. Taylor and M.A. Kaashoek concerning the relationship between ascent, descent, nullity, and defect for the case of linear operators remain valid in the context of linear relations, sometimes under the additional condition that the linear relation does not have any nontrivial singular chains. In particular, it is shown for a linear relation A with a trivial singular chain manifold whose ascent p is finite and whose nullity and defect are equal and finite that the linear space \mathfrak{S} is a direct sum of $\ker A^p$ and $\text{ran } A^p$. Furthermore it is shown that the various results which require the absence of singular chains are not valid when such chains are present.

© 2007 Elsevier Inc. All rights reserved.

AMS classification: Primary 47A05, 47A06; Secondary 15A04*Keywords:* Linear relation; Linear space; Ascent; Descent; Nullity; Defect; Singular chain; Complete reduction

* Corresponding author.

E-mail addresses: adrian.sandovici@allstudio.ro (A. Sandovici), desnoo@math.rug.nl (H. de Snoo), winkler@math.tu-berlin.de (H. Winkler).

¹ Supported by the “Fond zur Förderung der wissenschaftlichen Forschung” (FWF, Austria), grant number P15540-N05.

1. Introduction

Let A be a linear operator in a linear space \mathfrak{H} , which is not necessarily everywhere defined. Its kernel and range are denoted by $\ker A$ and $\operatorname{ran} A$. It is well known that

$$\ker A^n \subset \ker A^{n+1} \quad \text{and} \quad \operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n$$

for all $n \in \mathbb{N} \cup \{0\}$. The smallest nonnegative integer for which there is equality is called the ascent of A and the descent of A , denoted by $\alpha(A)$ and $\delta(A)$, respectively. In case no such number exists the ascent or descent of A is said to be infinite. The nullity and the defect of a linear operator A are defined by

$$n(A) = \dim \ker A, \quad d(A) = \dim \mathfrak{H} / \operatorname{ran} A.$$

For a linear operator A the quantities $\alpha(A)$ and $\delta(A)$ were introduced by Riesz [18] in connection with his investigation of compact linear operators, while the quantities $n(A)$ and $d(A)$ appear in [10,14] in connection with the perturbation theory of linear operators in Banach spaces. Heuser [11] considered these notions for a linear operator A in a linear space \mathfrak{H} under the condition that A is defined everywhere. The last condition was lifted by Taylor [21], whose treatment was completed by Kaashoek [12]. Kaashoek also provided a unified way of proving Taylor's results (see also [20,22]).

The concept of a linear relation in a linear space generalizes the one of a linear operator to that of a multivalued operator. A systematic treatment was given by Arens [1] and by Coddington [6]. This concept has been studied in a large number of papers, cf. [7]. It has proved to be useful in different areas, such as the extension theory of linear operators in spaces endowed with a metric, cf. [8], and the theory of degenerate differential equations and degenerate operator semigroups, cf. [2,9]. Recently the authors have given a structure theorem for linear relations in a finite-dimensional Euclidean space, cf. [19]. The methods used to develop such a structure theorem involve the notions of ascent, descent, nullity, and defect for linear relations.

The purpose of the present paper is to give a systematic treatment of these notions in the context of linear relations in linear spaces. It turns out that many of the results of Taylor and Kaashoek for linear operators remain valid in the context of linear relations. However, certain of their results are valid in the context of linear relations only under the additional condition that the linear relation does not have a nontrivial singular chain manifold (this notion is introduced below, see also [19]). Without this restriction, the original results cannot be carried over as simple examples show. The proofs in the present paper differ considerably from those of Taylor. In fact, the isomorphism results later employed by Kaashoek [12] for the operator case remain valid in the context of linear relations (sometimes with the restriction of a trivial singular chain manifold). In particular, it is shown for a linear relation A with a trivial singular chain manifold that the conditions $p = \alpha(A) < \infty$ and $n(A) = d(A) < \infty$ imply that $\alpha(A) = \delta(A)$ and that the linear space \mathfrak{H} is a direct sum of $\ker A^p$ and $\operatorname{ran} A^p$. This result completes the circle of ideas starting for linear operators A with $\operatorname{dom} A = \mathfrak{H}$ or $p = 1$ (cf. [21]) and the removal of these conditions in [12]. Furthermore it is shown when a linear relation A can be written as an operator-like sum of a relation whose kernel is trivial and whose range is the whole space and an everywhere defined operator with finite-dimensional range. The considerations in this paper are entirely algebraic.

The notions of ascent, descent, nullity, and defect have been systematically used in the seventies and eighties as tools in the study of several spectral properties of some classes of linear operators in Banach spaces, see for instance [3–5,12,13,17] and the references therein. In particular, Labrousse [15] introduced linear operators of quasi-Fredholm type and obtained an extension of the Kato decomposition for such operators (cf. [14]). It is possible to also consider such results in the

context of linear relations in normed linear spaces, cf. [16]. Furthermore, many of the present results can be used in the study of perturbations of linear relations in normed spaces, as for example the stability of semi-Fredholm relations under various perturbations.

A brief outline of the paper follows. To make the paper easily accessible the exposition is made self-contained. Some basic results from the theory of linear spaces due to Taylor and Kaashoek are recalled in Section 2. In Section 3 some general facts concerning linear relations in a linear space \mathfrak{H} are introduced. Section 4 presents isomorphism type results for the domain, range, kernel, and multivalued part of a nonnegative power of a linear relation (see [12] for the operator case). In Section 5 some basic results concerning ascent, descent, nullity, and defect of a relation are proved. Section 6 brings more and deeper information about these notions. Shifted linear relations are presented in Section 7. Completely reduced relations are defined and investigated in Section 8. The decomposition of a relation A as an operator-like sum of a suitable everywhere defined operator and a relation which has some ‘nice’ properties can be found in Section 9. Finally, in Section 10 it is shown that the results where the absence of a nontrivial singular chain manifold is required are not valid without this condition. In each individual case a relevant example is presented. Furthermore, in this section there are some examples to show that the inclusion of relations does not result in corresponding inequalities for the ascent and descent.

2. Linear spaces and quotient spaces

All linear spaces in this paper are assumed to be over the field \mathbb{K} of real or complex numbers. For the convenience of the reader some auxiliary results concerning quotient spaces and complementary subspaces are recalled.

Let \mathfrak{M} and \mathfrak{N} be subspaces of a linear space \mathfrak{H} . In this paper a subspace is always assumed to be linear. The sum $\mathfrak{M} + \mathfrak{N}$ of \mathfrak{M} and \mathfrak{N} is given by

$$\mathfrak{M} + \mathfrak{N} = \{x + y : x \in \mathfrak{M}, y \in \mathfrak{N}\},$$

and it is the smallest subspace of \mathfrak{H} which contains \mathfrak{M} and \mathfrak{N} . The sum is called direct if, in addition, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. The subspaces \mathfrak{M} and \mathfrak{N} of a linear space \mathfrak{H} are called *complementary* (in \mathfrak{H}) if $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$ and $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. The notation $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$ is used to denote that the subspaces \mathfrak{M} and \mathfrak{N} are complementary. For every subspace \mathfrak{M} of a linear space \mathfrak{H} there exists a, not necessarily unique, complementary subspace \mathfrak{N} such that $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$.

Two linear spaces \mathfrak{H}_1 and \mathfrak{H}_2 are said to be *isomorphic*, denoted by $\mathfrak{H}_1 \cong \mathfrak{H}_2$, whenever there exists a one-to-one linear mapping from \mathfrak{H}_1 onto \mathfrak{H}_2 . If \mathfrak{M} is a subspace of \mathfrak{H} then

$$\mathfrak{H}/\mathfrak{M} \quad \text{or} \quad \frac{\mathfrak{H}}{\mathfrak{M}}$$

denotes the linear space of all cosets $[x] = x + \mathfrak{M}$, with $x \in \mathfrak{H}$. Note that if for some subspaces \mathfrak{M} and \mathfrak{N} the relation $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{H}$ holds true, then $\mathfrak{H}/\mathfrak{M}$ and \mathfrak{N} are isomorphic as linear spaces. If \mathfrak{H} is a linear space, the dimension of \mathfrak{H} is denoted by $\dim \mathfrak{H}$. Isomorphic linear spaces have the same dimension. Note that if $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, then $\dim \mathfrak{H} = \dim \mathfrak{M} + \dim \mathfrak{N}$. In particular, if \mathfrak{M} is a subspace of a linear space \mathfrak{H} , then

$$\dim \mathfrak{H} = \dim \mathfrak{H}/\mathfrak{M} + \dim \mathfrak{M}.$$

Now let \mathfrak{M} and \mathfrak{N} be two subspaces of a linear space \mathfrak{H} . Then $\dim \mathfrak{M} \leq \dim \mathfrak{N}$ if for every $p \in \mathbb{N} \cup \{0\}$ the existence of p linearly independent vectors in \mathfrak{M} implies the existence of p linearly independent vectors in \mathfrak{N} .

The following two lemmas, together with their proofs, can be found in [21].

Lemma 2.1. *Let \mathfrak{M} and \mathfrak{N} be subspaces of a linear space \mathfrak{S} and assume that $\mathfrak{M} \subset \mathfrak{N}$. Then*

$$\dim \mathfrak{S}/\mathfrak{M} = \dim \mathfrak{S}/\mathfrak{N} + \dim \mathfrak{N}/\mathfrak{M}.$$

Lemma 2.2. *Let \mathfrak{M} and \mathfrak{N} be subspaces of \mathfrak{S} . Assume that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ and $\dim \mathfrak{S}/\mathfrak{M} \leq \dim \mathfrak{N} < \infty$. Then $\mathfrak{S} = \mathfrak{N} \oplus \mathfrak{M}$.*

The next three lemmas, together with their proofs, can be found in [12].

Lemma 2.3. *Let \mathfrak{M} and \mathfrak{N} be subspaces of a linear space \mathfrak{S} . Then*

$$\frac{\mathfrak{M}}{\mathfrak{M} \cap \mathfrak{N}} \cong \frac{\mathfrak{M} + \mathfrak{N}}{\mathfrak{N}}.$$

Lemma 2.4. *Let \mathfrak{M}_1 , \mathfrak{M}_2 , and \mathfrak{N} be subspaces of a linear space \mathfrak{S} and assume that $\mathfrak{M}_1 \subset \mathfrak{M}_2$. Then*

$$\dim \frac{\mathfrak{M}_1}{\mathfrak{M}_1 \cap \mathfrak{N}} \leq \dim \frac{\mathfrak{M}_2}{\mathfrak{M}_2 \cap \mathfrak{N}}.$$

Lemma 2.5. *Let \mathfrak{M}_1 , \mathfrak{M}_2 , and \mathfrak{N} be subspaces of a linear space \mathfrak{S} and assume that $\mathfrak{M}_1 \subset \mathfrak{M}_2$ and*

$$\dim \frac{\mathfrak{M}_1}{\mathfrak{M}_1 \cap \mathfrak{N}} = \dim \frac{\mathfrak{M}_2}{\mathfrak{M}_2 \cap \mathfrak{N}} < \infty.$$

Then $\mathfrak{M}_1 + \mathfrak{N} = \mathfrak{M}_2 + \mathfrak{N}$.

3. Linear relations in linear spaces

This section contains the definitions of most of the objects which will be studied in this paper. A *linear relation*, or relation for short, in a linear space \mathfrak{S} is a (linear) subspace of the space $\mathfrak{S} \times \mathfrak{S}$, the Cartesian product of \mathfrak{S} with itself.

3.1. Elementary definitions

The notations $\text{dom } A$ and $\text{ran } A$ for a linear relation A denote the domain and the range of A , defined by

$$\text{dom } A = \{x : \{x, y\} \in A\}, \quad \text{ran } A = \{y : \{x, y\} \in A\}.$$

Furthermore, $\ker A$ and $\text{mul } A$ denote the kernel and the multivalued part of A , defined by

$$\ker A = \{x : \{x, 0\} \in A\}, \quad \text{mul } A = \{y : \{0, y\} \in A\}.$$

A relation A is the graph of an operator if and only if $\text{mul } A = \{0\}$. The inverse A^{-1} is given by $\{\{y, x\} : \{x, y\} \in A\}$. The following identities express the duality of A and its inverse A^{-1} :

$$\text{dom } A^{-1} = \text{ran } A, \quad \text{ran } A^{-1} = \text{dom } A, \quad \ker A^{-1} = \text{mul } A, \quad \text{mul } A^{-1} = \ker A.$$

For relations A and B in a linear space \mathfrak{S} the *operator-like sum* $A + B$ is the relation in \mathfrak{S} defined by

$$A + B = \{\{x, y + z\} : \{x, y\} \in A, \{x, z\} \in B\},$$

and the *component-wise sum* $A \hat{+} B$ is the relation in \mathfrak{H} defined by

$$A \hat{+} B = \{\{x + u, y + v\} : \{x, y\} \in A, \{u, v\} \in B\};$$

this last sum is direct when $A \cap B = \{\{0, 0\}\}$. For $\lambda \in \mathbb{K}$ the relation λA in \mathfrak{H} is defined by

$$\lambda A = \{\{x, \lambda y\} : \{x, y\} \in A\},$$

while $A - \lambda$ stands for $A - \lambda I$, where I is the identity operator on \mathfrak{H} :

$$A - \lambda = \{\{x, y - \lambda x\} : \{x, y\} \in A\}.$$

For relations A and B in a linear space \mathfrak{H} the *product* AB is defined as the relation

$$AB = \{\{x, y\} : \{x, z\} \in B, \{z, y\} \in A \text{ for some } z \in \mathfrak{H}\}.$$

The product of relations is clearly associative. Hence A^n , $n \in \mathbb{Z}$, is defined as usual with $A^0 = I$ and $A^1 = A$. It is easily seen that

$$(A^{-1})^n = (A^n)^{-1}, \quad n \in \mathbb{Z}.$$

It is useful to observe that if A and B are relations in the same linear space \mathfrak{H} such that $A \subset B$, then also $A^{-1} \subset B^{-1}$. Hence, clearly, for all $n \in \mathbb{Z}$ it follows that $A^n \subset B^n$. In particular, the inclusion $A \subset B$ implies that for all $n \in \mathbb{Z}$

$$\begin{aligned} \text{dom } A^n &\subset \text{dom } B^n, & \text{ran } A^n &\subset \text{ran } B^n, \\ \ker A^n &\subset \ker B^n, & \text{mul } A^n &\subset \text{mul } B^n. \end{aligned}$$

Let \mathfrak{M} be a subspace of \mathfrak{H} . Then the restriction $A_{\mathfrak{M}}$ of a relation A in \mathfrak{H} is the following subrelation of A defined by:

$$A_{\mathfrak{M}} = \{\{x, y\} \in A : x, y \in \mathfrak{M}\}.$$

Note that $\text{ran } A_{\mathfrak{M}} \subset \mathfrak{M}$ and $\text{dom } A_{\mathfrak{M}} \subset \mathfrak{M}$ by definition. A subspace \mathfrak{M} is called *exactly range invariant under* A if $\text{ran } A_{\mathfrak{M}} = \mathfrak{M}$ and *exactly domain invariant under* A if $\text{dom } A_{\mathfrak{M}} = \mathfrak{M}$, respectively.

3.2. Singular chains

An important role is played by certain root manifolds of a relation A in a linear space \mathfrak{H} . The *root manifold* $\mathfrak{R}_0(A)$ is defined by

$$\mathfrak{R}_0(A) = \bigcup_{i=1}^{\infty} \ker A^i. \quad (3.1)$$

Similarly, the *root manifold* $\mathfrak{R}_{\infty}(A)$, is defined by

$$\mathfrak{R}_{\infty}(A) = \bigcup_{i=1}^{\infty} \text{mul } A^i. \quad (3.2)$$

Clearly the root manifolds $\mathfrak{R}_0(A)$ and $\mathfrak{R}_{\infty}(A)$ are subspaces of $\text{dom } A \subset \mathfrak{H}$ and $\text{ran } A \subset \mathfrak{H}$, respectively. The *singular chain manifold* $\mathfrak{R}_c(A)$ is defined as the intersection of the root manifolds $\mathfrak{R}_0(A)$ in (3.1) and $\mathfrak{R}_{\infty}(A)$ in (3.2):

$$\mathfrak{R}_c(A) = \mathfrak{R}_0(A) \cap \mathfrak{R}_{\infty}(A). \quad (3.3)$$

The linear space $\mathfrak{R}_c(A)$ is nontrivial if and only if there exists a number $s \in \mathbb{N}$ and elements $x_i \in \mathfrak{H}$, $1 \leq i \leq s$, not all equal to zero, such that

$$\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_{s-1}, x_s\}, \{x_s, 0\} \in A, \quad (3.4)$$

cf. [19]. A chain of the form (3.4) is said to be a *singular chain*. Without loss of generality a nontrivial singular chain of the form (3.4) may be replaced by a possibly shorter singular chain in which all elements x_i are nonzero. Clearly, if $\mathfrak{R}_c(A) = \{0\}$ and $\{x, y\} \in A^{n+m}$ with $n, m \in \mathbb{N}$, then there is a unique vector $z \in \mathfrak{H}$ such that $\{x, z\} \in A^n$ and $\{z, y\} \in A^m$. The root manifolds have some invariance properties which follow immediately from the definition:

$$\mathfrak{R}_0(A) = \mathfrak{R}_\infty(A^{-1}), \quad \mathfrak{R}_\infty(A) = \mathfrak{R}_0(A^{-1}). \quad (3.5)$$

Furthermore, it follows from (3.5) that

$$\mathfrak{R}_c(A) = \mathfrak{R}_0(A) \cap \mathfrak{R}_\infty(A) = \mathfrak{R}_\infty(A^{-1}) \cap \mathfrak{R}_0(A^{-1}) = \mathfrak{R}_c(A^{-1}).$$

The following result is sometimes useful.

Lemma 3.1. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. Let \mathfrak{M} be a subspace of \mathfrak{H} , then $\mathfrak{R}_c(A_{\mathfrak{M}}) = \{0\}$.*

Proof. Assume that $\mathfrak{R}_c(A_{\mathfrak{M}}) \neq \{0\}$, i.e. there exist $0 \neq x_i \in \mathfrak{H}$, $1 \leq i \leq n$, such that $\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_n, 0\} \in A_{\mathfrak{M}} \subset A$, which implies that $\mathfrak{R}_c(A) \neq \{0\}$, a contradiction. Hence $\mathfrak{R}_c(A_{\mathfrak{M}}) = \{0\}$. \square

3.3. Some useful observations

Some useful preparatory material will now be developed.

Lemma 3.2. *Let A be a relation in a linear space \mathfrak{H} . Then for all $n, m \in \mathbb{N} \cup \{0\}$*

$$\text{dom } A^{n+m} \subset \text{dom } A^n, \quad \text{ran } A^{n+m} \subset \text{ran } A^n, \quad (3.6)$$

$$\ker A^{n+m} \supset \ker A^n, \quad \text{mul } A^{n+m} \supset \text{mul } A^n, \quad (3.7)$$

and for all $p, k \in \mathbb{N} \cup \{0\}$

$$\ker A^p \subset \text{dom } A^k, \quad \text{mul } A^p \subset \text{ran } A^k. \quad (3.8)$$

Proof. Assume that $x \in \text{dom } A^{n+m}$, so that $\{x, y\} \in A^{n+m}$ for some $y \in \mathfrak{H}$. Since $A^{n+m} = A^m A^n$, it follows that $\{x, z\} \in A^n$ and $\{z, y\} \in A^m$ for some $z \in \mathfrak{H}$, which shows that $x \in \text{dom } A^n$. Therefore, $\text{dom } A^{n+m} \subset \text{dom } A^n$, and the first inclusion in (3.6) is proved.

In order to prove the first inclusion in (3.7) assume that $x \in \ker A^n$, so that $\{x, 0\} \in A^n$. Since $\{0, 0\} \in A^m$ as A^m is a subspace of $\mathfrak{H} \times \mathfrak{H}$, it follows that $\{x, 0\} \in A^{n+m}$, which shows that $x \in \ker A^{n+m}$. Therefore, $\ker A^{n+m} \supset \ker A^n$.

If $p \leq k$ then it follows from (3.7) that $\ker A^p \subset \ker A^k$ and since $\ker A^k \subset \text{dom } A^k$, it follows that in this case (3.8) holds true. Assume now that $p > k$ and let $x \in \ker A^p$, so that $\{x, 0\} \in A^p = A^{p-k} A^k$. Thus $\{x, y\} \in A^k$ and $\{y, 0\} \in A^{p-k}$, which shows that $x \in \text{dom } A^k$. Therefore the first inclusion in (3.8) is proved.

The remaining inclusions in (3.6)–(3.8) follow from the duality of A and A^{-1} . \square

Lemma 3.3. *Let A be a relation in a linear space \mathfrak{H} . If one of the following conditions*

$$\text{dom } A^r \cap \text{mul } A = \{0\} \quad \text{or} \quad \text{ran } A^r \cap \ker A = \{0\}$$

is satisfied for some $r \in \mathbb{N} \cup \{0\}$, then $\mathfrak{R}_c(A) = \{0\}$.

Proof. Assume that $\text{dom } A^r \cap \text{mul } A = \{0\}$. If $r = 0$ then $\text{mul } A = \{0\}$ and hence $\mathfrak{R}_c(A) = \{0\}$. Now let $r \in \mathbb{N}$. If $\mathfrak{R}_c(A) \neq \{0\}$, then A has a nontrivial singular chain of the form

$$\{0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, 0\}$$

for some non-zero vectors $x_i \in \mathfrak{H}$, $1 \leq i \leq n$. Clearly, $x_1 \in \text{mul } A$ and $x_1 \in \ker A^n \subset \text{dom } A^r$ by (3.8). Therefore,

$$x_1 \in \text{mul } A \cap \text{dom } A^r = \{0\},$$

which shows that $x_1 = 0$. This contradiction implies that $\mathfrak{R}_c(A) = \{0\}$. The argument for the other case is completely similar. \square

Lemma 3.4. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *If $\ker A^k = \ker A^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $\ker A^n = \ker A^k$ for all nonnegative integers $n \geq k$.*
- (ii) *If $\text{mul } A^k = \text{mul } A^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $\text{mul } A^n = \text{mul } A^k$ for all nonnegative integers $n \geq k$.*

Proof. (i) Assume that $\ker A^{n+1} = \ker A^n$. It will be shown that $\ker A^{n+2} = \ker A^{n+1}$, and then, the statement will follow by induction. Clearly, (3.7) shows that $\ker A^{n+1} \subset \ker A^{n+2}$, so that only the converse inclusion remains to be proved. Let $x \in \ker A^{n+2}$, so that $\{x, 0\} \in A^{n+2} = A^{n+1}A$. Thus, $\{x, y\} \in A$ and $\{y, 0\} \in A^{n+1}$ for some $y \in \mathfrak{H}$. Now $y \in \ker A^{n+1} = \ker A^n$ by the induction hypothesis, which shows that $\{y, 0\} \in A^n$. Therefore $\{x, 0\} \in A^{n+1}$, so that $x \in \ker A^{n+1}$, which implies (i).

The statement (ii) follows from the statement in (i) due to the duality of A and A^{-1} . \square

Lemma 3.5. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *If $\text{dom } A^k = \text{dom } A^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $\text{dom } A^n = \text{dom } A^k$ for all nonnegative integers $n \geq k$.*
- (ii) *If $\text{ran } A^k = \text{ran } A^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $\text{ran } A^n = \text{ran } A^k$ for all nonnegative integers $n \geq k$.*

Proof. (i) Assume that $\text{dom } A^n = \text{dom } A^{n+1}$. It suffices to show that $\text{dom } A^{n+1} = \text{dom } A^{n+2}$. Clearly, (3.6) shows that $\text{dom } A^{n+2} \subset \text{dom } A^{n+1}$, so that only the converse inclusion remains to be proved. Assume that $x \in \text{dom } A^{n+1}$, so that $\{x, y\} \in A^{n+1} = A^n A$. Hence, $\{x, z\} \in A$ and $\{z, y\} \in A^n$ for some $z \in \mathfrak{H}$. Now $z \in \text{dom } A^n = \text{dom } A^{n+1}$ implies that $\{z, u\} \in A^{n+1}$ for some $u \in \mathfrak{H}$. This leads to $\{x, u\} \in A^{n+2}$ so that $x \in \text{dom } A^{n+2}$. Hence $\text{dom } A^{n+1} \subset \text{dom } A^{n+2}$.

The statement (ii) follows from the statement in (i) due to the duality of A and A^{-1} . \square

Corollary 3.6. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *$\text{dom } A = \mathfrak{H}$ if and only if $\text{dom } A^p = \mathfrak{H}$ for some (and hence for all) $p \in \mathbb{N}$;*
- (ii) *$\text{ran } A = \mathfrak{H}$ if and only if $\text{ran } A^p = \mathfrak{H}$ for some (and hence for all) $p \in \mathbb{N}$.*

Proof. (i) If $\text{dom } A = \mathfrak{H}$, then it follows from Lemma 3.5 that $\text{dom } A^p = \mathfrak{H}$ for all $p \in \mathbb{N}$. Conversely, if $\text{dom } A^p = \mathfrak{H}$ for some $p \in \mathbb{N}$, then clearly $\text{dom } A^i = \mathfrak{H}$ for $1 \leq i \leq p$, due to (3.6). The proof of (ii) is analogous. \square

3.4. Nullity, defect, ascent, and descent

The statements in Lemmas 3.2 and 3.4 lead to the introduction of the *ascent* and the *coascent* of A by

$$\begin{aligned}\alpha(A) &= \min\{k \in \mathbb{N} : \ker A^k = \ker A^{k+1}\}, \\ \alpha_c(A) &= \min\{k \in \mathbb{N} : \text{mul } A^k = \text{mul } A^{k+1}\},\end{aligned}$$

respectively, whenever these minima exist. If no such numbers exist the ascent and coascent of A are defined to be ∞ . Clearly,

$$\alpha(A) = \alpha_c(A^{-1}),$$

so that the notions of ascent and coascent preserve the duality of A and A^{-1} . Likewise, the statements in Lemmas 3.2 and 3.5 lead to the introduction of the *descent* and the *codescent* of A by

$$\begin{aligned}\delta(A) &= \min\{k \in \mathbb{N} : \text{ran } A^k = \text{ran } A^{k+1}\}, \\ \delta_c(A) &= \min\{k \in \mathbb{N} : \text{dom } A^k = \text{dom } A^{k+1}\},\end{aligned}$$

respectively, whenever these minima exist. If no such numbers exist the descent and codescent of A are defined to be ∞ . Clearly,

$$\delta(A) = \delta_c(A^{-1}),$$

so that the notions of descent and codescent preserve the duality of A and A^{-1} . Observe that $\alpha(A) = 0$ ($\alpha_c(A) = 0$) if and only if $\ker A = \{0\}$ ($\text{mul } A = \{0\}$) and that $\delta(A) = 0$ ($\delta_c(A) = 0$) if and only if $\text{ran } A = \mathfrak{H}$ ($\text{dom } A = \mathfrak{H}$).

Furthermore, define the *nullity* and the *conullity* of A by

$$n(A) = \dim \ker A, \quad n_c(A) = \dim \text{mul } A,$$

and define the *defect* (deficiency in, for instance, [14]) and the *codefect* of A by

$$d(A) = \dim \mathfrak{H} / \text{ran } A, \quad d_c(A) = \dim \mathfrak{H} / \text{dom } A.$$

Note that the nullity and conullity, defect and codefect of a linear relation are not necessarily finite either (in that case they are defined as ∞), and that the following relation

$$n(A) = n_c(A^{-1}), \quad d(A) = d_c(A^{-1}),$$

shows that the notions of nullity and conullity, and defect and codefect preserve the duality of A and A^{-1} .

Note that $n(A) = 0$ ($n_c(A) = 0$) and $d(A) = 0$ ($d_c(A) = 0$) are also equivalent with $\ker A = \{0\}$ ($\text{mul } A = \{0\}$) and $\text{ran } A = \mathfrak{H}$ ($\text{dom } A = \mathfrak{H}$), respectively. In the following, results for the ascent, descent, nullity, and defect of a relation will be obtained. Results for their counterparts may be obtained by considering the inverse of the relation.

4. Isomorphism type results for linear relations

This section contains some isomorphism type results in the context of relations in linear spaces. For the sake of completeness these results are stated in pairs (one result related to its companion by going from a relation to its inverse). Note that only in the last result there is a condition concerning the singular chain manifold; the other results are valid without this condition.

Lemma 4.1. *Let A be a relation in a linear space \mathfrak{S} and let $i, k \in \mathbb{N} \cup \{0\}$. Then*

$$\frac{\text{dom } A^i}{(\text{ran } A^k + \ker A^i) \cap \text{dom } A^i} \cong \frac{\text{ran } A^i}{\text{ran } A^{i+k}}, \quad (4.1)$$

and

$$\frac{\text{ran } A^i}{(\text{dom } A^k + \text{mul } A^i) \cap \text{ran } A^i} \cong \frac{\text{dom } A^i}{\text{dom } A^{i+k}}. \quad (4.2)$$

Proof. Let J be the linear relation from $\text{dom } A^i$ to $\text{ran } A^i / \text{ran } A^{i+k}$ defined by

$$J = \{\{x, [x']\} : \{x, x'\} \in A^i\},$$

where $[x']$ denotes the equivalence class in the quotient space $\text{ran } A^i / \text{ran } A^{i+k}$ to which $x' \in \text{ran } A^i$ belongs. Actually J is (the graph of) an operator. To see this, let $m \in \text{mul } J$, i.e., $\{0, m\} \in J$. Then $m = [x']$ for some $x' \in \text{mul } A^i$. Thus $x' \in \text{mul } A^i \subset \text{mul } A^{i+k} \subset \text{ran } A^{i+k}$, so that $m = [x'] = [0]$. Hence J is an operator from $\text{dom } A^i$ to $\text{ran } A^i / \text{ran } A^{i+k}$, given by $Jx = [x']$ if $\{x, x'\} \in A^i$. Clearly, J is surjective. Next it is shown that

$$\ker J = (\text{ran } A^k + \ker A^i) \cap \text{dom } A^i,$$

which implies that J induces an isomorphism between the spaces in (4.1).

Assume that $x \in \ker J$, so that $Jx = [x']$ for some $\{x, x'\} \in A^i$ with $x' \in \text{ran } A^{i+k}$. Then $\{y, x'\} \in A^{i+k}$ for some $y \in \text{dom } A^{i+k}$ which implies that there exists $z \in \mathfrak{S}$ such that $\{y, z\} \in A^k$ and $\{z, x'\} \in A^i$. It follows that

$$\{x - z, 0\} = \{x, x'\} - \{z, x'\} \in A^i,$$

so that $x - z \in \ker A^i$, which shows that

$$x = z + (x - z) \in \text{ran } A^k + \ker A^i.$$

Since $x \in \text{dom } A^i$ it follows that

$$\ker J \subset (\text{ran } A^k + \ker A^i) \cap \text{dom } A^i.$$

To show the converse inclusion, let $x \in (\text{ran } A^k + \ker A^i) \cap \text{dom } A^i$, so that $\{x, x'\} \in A^i$ for some $x' \in \text{ran } A^i$. Then $x = y + z$ with $y \in \text{ran } A^k$ and $z \in \ker A^i$, which implies that

$$\{y, x'\} = \{x, x'\} - \{z, 0\} \in A^i.$$

Since $\{w, y\} \in A^k$ for some $w \in \text{dom } A^k$ and $\{y, x'\} \in A^i$, it follows that $\{w, x'\} \in A^{i+k}$ and hence $x' \in \text{ran } A^{i+k}$. Therefore $[x'] = [0]$ which shows that $x \in \ker J$.

Finally, (4.2) follows from (4.1) with A^{-1} instead of A . \square

Lemma 4.2. *Let A be a relation in a linear space \mathfrak{S} and let $i, k \in \mathbb{N} \cup \{0\}$. Then*

$$\frac{\ker A^{i+k}}{(\ker A^i + \text{ran } A^k) \cap \ker A^{i+k}} \cong \frac{\ker A^k \cap \text{ran } A^i}{\ker A^k \cap \text{ran } A^{i+k}}, \quad (4.3)$$

and

$$\frac{\text{mul } A^{i+k}}{(\text{mul } A^i + \text{dom } A^k) \cap \text{mul } A^{i+k}} \cong \frac{\text{mul } A^k \cap \text{dom } A^i}{\text{mul } A^k \cap \text{dom } A^{i+k}}. \quad (4.4)$$

Proof. Observe that $\{x, 0\} \in A^{i+k} = A^k A^i$ implies that $\{x, x'\} \in A^i$ and $\{x', 0\} \in A^k$, so that $x' \in \ker A^k \cap \text{ran } A^i$. Denote by $[x']$ the equivalence class of x' relative to the quotient space

$$(\ker A^k \cap \text{ran } A^i) / (\ker A^k \cap \text{ran } A^{i+k}). \quad (4.5)$$

Hence

$$J = \{\{x, [x']\} : \{x, x'\} \in A^i, \{x', 0\} \in A^k\}$$

defines a linear relation from $\ker A^{i+k}$ to the quotient space in (4.5). Actually J is (the graph of) a linear operator. To see this, let $m \in \text{mul } J$, i.e., $\{0, m\} \in J$. Then $m = [x']$ for some $x' \in \ker A^k \cap \text{mul } A^i$. Since $\text{mul } A^i \subset \text{mul } A^{i+k} \subset \text{ran } A^{i+k}$ it follows that $x' \in \ker A^k \cap \text{ran } A^{i+k}$, so that $m = [x'] = [0]$, which shows that J is an operator from $\ker A^{i+k}$ to the quotient space in (4.5). Clearly, J is surjective. Next it is shown that

$$\ker J = (\ker A^i + \text{ran } A^k) \cap \ker A^{i+k}, \quad (4.6)$$

which implies that J induces an isomorphism between the spaces in (4.3).

Let $x \in \ker J$ so that $\{x, x'\} \in A^i$, $\{x', 0\} \in A^k$ for some $x' \in \text{ran } A^{i+k}$. Hence $\{z, x'\} \in A^{i+k}$ for some $z \in \text{dom } A^{i+k}$, and therefore $\{z, w\} \in A^k$ and $\{w, x'\} \in A^i$ for some $w \in \text{ran } A^k \cap \text{dom } A^i$. Clearly,

$$\{x - w, 0\} = \{x, x'\} - \{w, x'\} \in A^i,$$

which shows that $x - w \in \ker A^i$ and hence

$$x = (x - w) + w \in \ker A^i + \text{ran } A^k,$$

so that $\ker J \subset (\ker A^i + \text{ran } A^k) \cap \ker A^{i+k}$. To show the converse inclusion, assume that $x \in (\ker A^i + \text{ran } A^k) \cap \ker A^{i+k}$, so that $x = y + z$ for some $y \in \ker A^i$ and $z \in \text{ran } A^k$. If $Jx = [x']$ then $\{x, x'\} \in A^i$ and $\{x', 0\} \in A^k$, and then

$$\{z, x'\} = \{x, x'\} - \{y, 0\} \in A^i.$$

It follows from $\{z, x'\} \in A^i$ and $\{w, z\} \in A^k$ that $\{w, x'\} \in A^{i+k}$, so that $x' \in \text{ran } A^{i+k}$ and then $Jx = [x'] = [0]$. Therefore $(\ker A^i + \text{ran } A^k) \cap \ker A^{i+k} \subset \ker J$, so that (4.6) holds true.

Finally, (4.4) follows from (4.3) with A^{-1} instead of A . \square

Lemma 4.3. Let A be a relation in a linear space \mathfrak{H} and let $i \in \mathbb{N} \cup \{0\}$. Then

$$\dim \frac{\ker A}{\ker A \cap \text{ran } A^i} = \dim \frac{\ker A^i}{\text{ran } A \cap \ker A^i}, \quad (4.7)$$

and

$$\dim \frac{\text{mul } A}{\text{mul } A \cap \text{dom } A^i} = \dim \frac{\text{mul } A^i}{\text{dom } A \cap \text{mul } A^i}. \quad (4.8)$$

Proof. It follows from Lemma 4.2 with $k = 1$ that

$$\frac{\ker A^{i+1}}{(\ker A^i + \text{ran } A) \cap \ker A^{i+1}} \cong \frac{\ker A \cap \text{ran } A^i}{\ker A \cap \text{ran } A^{i+1}}, \quad i \in \mathbb{N} \cup \{0\},$$

and it follows from Lemma 2.3 with $\mathfrak{M} = \ker A^{i+1}$ and $\mathfrak{N} = \ker A^i + \text{ran } A$, that

$$\frac{\ker A^{i+1}}{(\ker A^i + \text{ran } A) \cap \ker A^{i+1}} \cong \frac{\ker A^{i+1} + \ker A^i + \text{ran } A}{\ker A^i + \text{ran } A} = \frac{\ker A^{i+1} + \text{ran } A}{\ker A^i + \text{ran } A}.$$

Thus it follows that

$$\dim \frac{\ker A \cap \text{ran } A^i}{\ker A \cap \text{ran } A^{i+1}} = \dim \frac{\ker A^{i+1} + \text{ran } A}{\ker A^i + \text{ran } A}, \quad i \in \mathbb{N} \cup \{0\}. \quad (4.9)$$

Observe that a repeated application of Lemma 2.1 gives

$$\dim \frac{\ker A}{\ker A \cap \text{ran } A^i} = \sum_{h=0}^{i-1} \dim \frac{\ker A \cap \text{ran } A^h}{\ker A \cap \text{ran } A^{h+1}}, \quad (4.10)$$

and that a repeated application of Lemma 2.1 also gives

$$\dim \frac{\ker A^i + \text{ran } A}{\text{ran } A} = \sum_{h=0}^{i-1} \dim \frac{\ker A^{h+1} + \text{ran } A}{\ker A^h + \text{ran } A}. \quad (4.11)$$

A combination of (4.9)–(4.11) leads to (4.7).

Finally, (4.8) follows from (4.7) with A^{-1} instead of A . \square

Lemma 4.4. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$ and let $i, k \in \mathbb{N} \cup \{0\}$. Then*

$$\frac{\ker A^{i+k}}{\ker A^i} \cong \ker A^k \cap \text{ran } A^i, \quad (4.12)$$

and

$$\frac{\text{mul } A^{i+k}}{\text{mul } A^i} \cong \text{mul } A^k \cap \text{dom } A^i. \quad (4.13)$$

Proof. Let $x \in \ker A^{i+k}$, so that there exists a vector $x' \in \ker A^k$ such that $\{x, x'\} \in A^i$. Clearly, this vector x' is unique under the assumption $\mathfrak{R}_c(A) = \{0\}$. Observe that $x' \in \ker A^k \cap \text{ran } A^i$. Therefore

$$Jx = x'$$

defines a linear operator from $\ker A^{i+k}$ to $\ker A^k \cap \text{ran } A^i$. In order to show that J is surjective, let $x' \in \ker A^k \cap \text{ran } A^i$. Then $\{x', 0\} \in A^k$ and $\{x, x'\} \in A^i$ for some $x \in \mathfrak{H}$. This implies that $x \in \ker A^{i+k}$ and $Jx = x'$. Hence, J maps onto $\ker A^k \cap \text{ran } A^i$. Since $\ker J = \ker A^i$, the mapping J induces an isomorphism between the quotient space $\ker A^{i+k} / \ker A^i$ and $\ker A^k \cap \text{ran } A^i$. Thus (4.12) is completely proved.

Finally, (4.13) follows from (4.12) with A^{-1} instead of A . \square

The proofs of the previous lemmas require the explicit construction of certain isomorphisms. The construction of these isomorphisms for the case of linear operators goes back to Kaashoek [12] and can indeed be extended to the case of linear relations.

5. Ascent, decent, nullity, and defect

This section contains some elementary results concerning the ascent, descent, nullity, and defect of a relation A in a linear space \mathfrak{H} .

5.1. Some results for nullity and defect

The following lemma is a preliminary result from which information concerning nullity and defect will follow.

Lemma 5.1. *Let A be a relation in a linear space \mathfrak{H} and let $r \in \mathbb{N} \cup \{0\}$. Then*

$$\dim \frac{\ker A^r + \text{ran } A}{\text{ran } A} = \dim \frac{\ker A^r}{\ker A^r \cap \text{ran } A} \leq \dim \frac{\text{dom } A^r}{\text{dom } A^r \cap \text{ran } A} = \dim \frac{\text{dom } A^r + \text{ran } A}{\text{ran } A}.$$

Proof. The first equality follows from Lemma 2.3 with $\mathfrak{M} = \ker A^r$ and $\mathfrak{N} = \text{ran } A$. The inequality follows from Lemma 2.4 with $\mathfrak{M}_1 = \ker A^r$, $\mathfrak{M}_2 = \text{dom } A^r$, and $\mathfrak{N} = \ker A$. The last equality follows from Lemma 2.3 with $\mathfrak{M} = \text{dom } A^r$ and $\mathfrak{N} = \text{ran } A$. \square

A direct consequence of Lemma 5.1 is the following result concerning the defect.

Lemma 5.2. *Let A be a relation in a linear space \mathfrak{H} and let $r \in \mathbb{N} \cup \{0\}$. If*

$$\text{dom } A^r + \text{ran } A = \mathfrak{H},$$

then

$$d(A) = \dim \frac{\text{dom } A^r}{\text{ran } A \cap \text{dom } A^r}.$$

In particular, if

$$\ker A^r + \text{ran } A = \mathfrak{H},$$

then

$$d(A) = \dim \frac{\text{dom } A^r}{\text{ran } A \cap \text{dom } A^r} = \dim \frac{\ker A^r}{\text{ran } A \cap \ker A^r}. \quad (5.1)$$

From this follows an important inequality between nullity and defect. The condition (5.2) will be explained below, see Lemma 5.5.

Lemma 5.3. *Let A be a relation in a linear space \mathfrak{H} and let $r \in \mathbb{N} \cup \{0\}$. If*

$$\ker A \cap \text{ran } A^r = \{0\}, \quad (5.2)$$

then

$$n(A) = \dim \frac{\ker A^r}{\text{ran } A \cap \ker A^r} \leq d(A). \quad (5.3)$$

If, in addition,

$$\ker A^r + \text{ran } A = \mathfrak{H}, \quad (5.4)$$

then there is equality in (5.3). If (5.2) holds and there is equality in (5.3) with $n(A) = d(A) < \infty$, then (5.4) holds.

Proof. The first conclusion follows from (4.7) (with $i = r$) and Lemma 2.4 with $\mathfrak{M}_1 = \ker A^r$, $\mathfrak{M}_2 = \mathfrak{H}$, and $\mathfrak{N} = \text{ran } A$:

$$n(A) = \dim \frac{\ker A}{\ker A \cap \text{ran } A^r} = \dim \frac{\ker A^r}{\text{ran } A \cap \ker A^r} \leq \dim \frac{\mathfrak{H}}{\text{ran } A \cap \mathfrak{H}} = d(A).$$

If (5.2) and (5.4) hold, then it follows from (5.1) and Lemma 2.3 that there is equality in (5.3). If (5.2) holds and there is equality in (5.3) with $n(A) = d(A) < \infty$, then

$$n(A) = \dim \frac{\ker A^r}{\operatorname{ran} A \cap \ker A^r} = \dim \frac{\mathfrak{H}}{\operatorname{ran} A \cap \mathfrak{H}} = d(A) < \infty.$$

It follows from Lemma 2.5 that $\operatorname{ran} A + \ker A^r = \operatorname{ran} A + \mathfrak{H} = \mathfrak{H}$. \square

The following result relates the nullity and the defect of a relation to that of its powers.

Lemma 5.4. *Let A be a relation in a linear space \mathfrak{H} and let $k \in \mathbb{N}$. Then:*

- (i) *If $n(A) < \infty$, then $n(A^k) \leq kn(A)$.*
- (ii) *If $d(A) < \infty$, then $d(A^k) \leq kd(A)$.*

Proof. (i) Let $n \geq 0$. Since $\ker A^n \subset \ker A^{n+1}$ it follows that there exists a complementary subspace \mathfrak{N} (relative to $\ker A^{n+1}$) such that $\ker A^{n+1} = \ker A^n \oplus \mathfrak{N}$. It will be shown that $\dim \mathfrak{N} \leq n(A)$. The case $\dim \mathfrak{N} = 0$ is trivial, hence assume that $\dim \mathfrak{N} > 0$. Let $x_1, x_2, \dots, x_p \in \mathfrak{N}$ be linearly independent, $1 \leq p \leq \dim \mathfrak{N}$. Then (because $\mathfrak{N} \subset \ker A^{n+1}$) there exist $y_1, y_2, \dots, y_p \in \ker A$ such that

$$\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_p, y_p\} \in A^n.$$

Assume that $\sum_{i=1}^p c_i y_i = 0$ for certain $c_i \in \mathbb{K}$, $1 \leq i \leq p$. Then

$$\sum_{i=1}^p c_i \{x_i, y_i\} = \left\{ \sum_{i=1}^p c_i x_i, \sum_{i=1}^p c_i y_i \right\} = \left\{ \sum_{i=1}^p c_i x_i, 0 \right\} \in A^n,$$

so that $\sum_{i=1}^p c_i x_i \in \ker A^n \cap \mathfrak{N}$. Since \mathfrak{N} and $\ker A^n$ are complementary spaces it follows that $\sum_{i=1}^p c_i x_i = 0$ which implies that $c_i = 0$, $1 \leq i \leq p$. This means that for any p linearly independent vectors in \mathfrak{N} there exist p linearly independent vectors in $\ker A$. Hence $\dim \mathfrak{N} \leq \dim \ker A = n(A)$. Thus $n(A^{n+1}) \leq n(A^n) + n(A)$, so that the statement follows by induction; recall that $n(A^0) = 0$.

(ii) Since $d(A^0) = 0$, the case $k = 0$ is trivial. Assume $k \in \mathbb{N}$ and define

$$\mathfrak{M}_k = \operatorname{ran} A^{k-1} / \operatorname{ran} A^k.$$

It follows from Lemma 3.2(i) and Lemma 2.1 that $d(A^k) = d(A^{k-1}) + \dim \mathfrak{M}_k$ and a repeated application gives

$$d(A^k) = \dim \mathfrak{M}_1 + \dim \mathfrak{M}_2 + \dots + \dim \mathfrak{M}_k, \quad k \in \mathbb{N}. \quad (5.5)$$

Note that $\dim \mathfrak{M}_1 = \dim(\operatorname{ran} A^0 / \operatorname{ran} A^1) = \dim(\mathfrak{H} / \operatorname{ran} A) = d(A)$. Now the inequality

$$\dim \mathfrak{M}_{n+1} \leq \dim \mathfrak{M}_n, \quad n \in \mathbb{N}, \quad (5.6)$$

will be shown. Let $[y_1], [y_2], \dots, [y_p] \in \mathfrak{M}_{n+1}$ be linearly independent cosets. Then $y_i \in \operatorname{ran} A^n$ for $1 \leq i \leq p$, so there exist $x_1, x_2, \dots, x_p \in \operatorname{ran} A^{n-1}$ such that $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_p, y_p\} \in A$ (even if $n = 1$ because $\operatorname{dom} A \subset \mathfrak{H} = \operatorname{ran} A^0$). Now if $\sum_{i=1}^p c_i [x_i] = [\sum_{i=1}^p c_i x_i] = [0]$ in \mathfrak{M}_n for certain $c_i \in \mathbb{K}$, then $\sum_{i=1}^p c_i x_i \in \operatorname{ran} A^n$, and hence $\sum_{i=1}^p c_i y_i \in \operatorname{ran} A^{n+1}$. It follows that

$$\sum_{i=1}^p c_i [y_i] = \left[\sum_{i=1}^p c_i y_i \right] = [0],$$

which implies $c_i = 0$, $1 \leq i \leq p$. Hence for any p linearly independent vectors in \mathfrak{M}_{n+1} there exist p linearly independent vectors in \mathfrak{M}_n . Therefore (5.6) has been established. The statement now follows from (5.5) and (5.6), since $\dim \mathfrak{M}_1 = d(A)$. \square

5.2. Some results for ascent and descent

The following lemma is very useful: it explains the absence of singular chains when the ascent is finite, cf. Lemma 3.3.

Lemma 5.5. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *If $\ker A \cap \operatorname{ran} A^p = \{0\}$ for some $p \in \mathbb{N} \cup \{0\}$, then $\mathfrak{R}_c(A) = \{0\}$ and $\alpha(A) \leq p$.*
- (ii) *If $\mathfrak{R}_c(A) = \{0\}$ and $\alpha(A) \leq p$ for some $p \in \mathbb{N} \cup \{0\}$, then $\ker A^k \cap \operatorname{ran} A^p = \{0\}$ for $k \in \mathbb{N}$.*

Proof. (i) Assume that $\ker A \cap \operatorname{ran} A^p = \{0\}$. By Lemma 3.3 it follows that $\mathfrak{R}_c(A) = \{0\}$. Let $x \in \ker A^{p+1}$. Then there exists $y \in \mathfrak{H}$ such that $\{x, y\} \in A^p$ and $\{y, 0\} \in A$. Thus $y \in \operatorname{ran} A^p \cap \ker A = \{0\}$, so that $y = 0$ and therefore $x \in \ker A^p$. Hence $\ker A^{p+1} \subset \ker A^p$. This shows that $\alpha(A) \leq p$.

(ii) Assume that $\alpha(A) \leq p$, from which it follows that $\ker A^{p+k} = \ker A^p$, and hence $\ker A^{p+k}/\ker A^p = \{0\}$. Due to the assumption $\mathfrak{R}_c(A) = \{0\}$ the identity (4.12) may be applied (with $i = p$) so that $\ker A^k \cap \operatorname{ran} A^p = \{0\}$. \square

Lemma 5.6. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *Assume that for some $p \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ there exists a subspace \mathfrak{M}_k of \mathfrak{H} such that*

$$\mathfrak{M}_k \subset \ker A^p, \quad \operatorname{dom} A^p = (\operatorname{dom} A^p \cap \operatorname{ran} A^k) + \mathfrak{M}_k. \quad (5.7)$$
Then $\delta(A) \leq p$.
- (ii) *Assume that $\delta(A) \leq p$ for some $p \in \mathbb{N} \cup \{0\}$. Then for every $k \in \mathbb{N}$ there exists a subspace \mathfrak{M}_k of \mathfrak{H} such that*

$$\mathfrak{M}_k \subset \ker A^p, \quad \mathfrak{M}_k \cap \operatorname{ran} A^k = \{0\}, \quad \operatorname{dom} A^p = (\operatorname{dom} A^p \cap \operatorname{ran} A^k) \oplus \mathfrak{M}_k. \quad (5.8)$$

Proof. (i) It will be shown that $\operatorname{ran} A^p \subset \operatorname{ran} A^{p+k}$. As the converse inclusion is obvious it follows that $\operatorname{ran} A^p = \operatorname{ran} A^{p+k}$, which implies that $\delta(A) \leq p$. Let $y \in \operatorname{ran} A^p$. Then $\{x, y\} \in A^p$ for some $x \in \operatorname{dom} A^p$. By (5.7), there exist $u \in \operatorname{ran} A^k$ and $v \in \mathfrak{M}_k \subset \ker A^p$ such that $x = u + v$. Thus,

$$\{u, y\} = \{x, y\} - \{v, 0\} \in A^p,$$

which implies that $y \in \operatorname{ran} A^{p+k}$. Hence $\operatorname{ran} A^p \subset \operatorname{ran} A^{p+k}$.

(ii) Let $\delta(A) \leq p$. For some fixed $k \geq 1$ choose a complementary subspace \mathfrak{N}_k (relative to $\operatorname{dom} A^p$), such that

$$\operatorname{dom} A^p = (\operatorname{dom} A^p \cap \operatorname{ran} A^k) \oplus \mathfrak{N}_k. \quad (5.9)$$

Let \mathbf{H} be a Hamel basis for \mathfrak{N}_k . Then for any vector $v \in \mathbf{H}$ there exists a vector v' such that $\{v, v'\} \in A^p$. Since $\operatorname{ran} A^p = \operatorname{ran} A^{p+k}$ it follows that there also exists some $w \in \operatorname{ran} A^k$ such that $\{w, v'\} \in A^p$. Then $\{v - w, 0\} = \{v, v'\} - \{w, v'\} \in A^p$. Since for each $v \in \mathbf{H}$ such a $w \in \operatorname{ran} A^k$ exists, let \mathfrak{M}_k be the linear space generated by these differences $v - w$. Clearly, $\mathfrak{M}_k \subset \ker A^p$.

Furthermore, $\mathfrak{M}_k \cap \text{ran } A^k = \{0\}$. To see this, let $y \in \mathfrak{M}_k \cap \text{ran } A^k$. Then $y = \sum_{i=1}^n c_i \times (v_i - w_i)$ for certain $c_i \in \mathbb{K}$, $v_i \in \mathbf{H}$, $w_i \in \text{ran } A^k$, $1 \leq i \leq n$, for some $n \in \mathbb{N}$. Since $\sum_{i=1}^n c_i v_i \in \mathfrak{N}_k$ and $y + \sum_{i=1}^n c_i w_i \in \text{ran } A^k$, it follows from (5.9) that

$$\sum_{i=1}^n c_i v_i = y + \sum_{i=1}^n c_i w_i \in \text{ran } A^k \cap \mathfrak{N}_k = \{0\}.$$

The vectors v_i are linearly independent, which leads to $c_i = 0$, $1 \leq i \leq n$, so that $y = 0$. Hence $\mathfrak{M}_k \cap \text{ran } A^k = \{0\}$.

Finally, the identity $\text{dom } A^p = (\text{dom } A^p \cap \text{ran } A^k) \oplus \mathfrak{M}_k$ has to be shown. Since $\mathfrak{M}_k \subset \ker A^p \subset \text{dom } A^p$ the inclusion $\text{dom } A^p \supset (\text{dom } A^p \cap \text{ran } A^k) \oplus \mathfrak{M}_k$ is clear. It remains to show the converse inclusion. Let any $x \in \text{dom } A^p$, so that $x = u + v$ for some $u \in \text{ran } A^k$ and $v \in \mathfrak{N}_k$. The vector v can be written as $v = \sum_{i=1}^n a_i v_i$ for certain $a_i \in \mathbb{K}$, $v_i \in \mathbf{H}$, $1 \leq i \leq n$, for some $n \in \mathbb{N}$. For every $v_i \in \mathbf{H}$ choose a vector $w_i \in \text{ran } A^k$ such that $v_i - w_i \in \mathfrak{M}_k$. Then

$$x = u + \sum_{i=1}^n a_i w_i + \sum_{i=1}^n a_i (v_i - w_i),$$

with $u + \sum_{i=1}^n a_i w_i \in \text{ran } A^k$ and $\sum_{i=1}^n a_i (v_i - w_i) \in \mathfrak{M}_k$.

Hence the constructed space \mathfrak{M}_k satisfies all conditions in (5.8), which completes the proof of (ii). \square

Theorem 5.7. *Let A be a relation in a linear space \mathfrak{H} .*

(i) *Assume that $\mathfrak{N}_c(A) = \{0\}$. If $\alpha(A) < \infty$ and $\delta(A) < \infty$, then*

$$\alpha(A) \leq \delta(A). \quad (5.10)$$

(ii) *Assume that $p = \alpha(A) < \infty$ and $q = \delta(A) < \infty$. If $p = q$ then*

$$\text{dom } A^p \subset \text{ran } A + \text{dom } A^q. \quad (5.11)$$

Moreover, if $p \leq q$ and (5.11) holds, then $p = q$.

Proof. (i) Assume that $\mathfrak{N}_c(A) = \{0\}$ and that $p = \alpha(A) < \infty$ and $q = \delta(A) < \infty$. If $p > q$, then by definition $\text{ran } A^p = \text{ran } A^q$ and in particular $\dim(\ker A \cap \text{ran } A^q) = \dim(\ker A \cap \text{ran } A^p)$. Due to the assumption $\mathfrak{N}_c(A) = \{0\}$ the identity (4.12) may be applied so that

$$\dim \frac{\ker A^{q+1}}{\ker A^q} = \dim \frac{\ker A^{p+1}}{\ker A^p} = 0$$

(where the last identity follows from $p = \alpha(A)$) which implies that $p \leq q$. This contradiction shows that (5.10) is valid.

(ii) Assume that $p = \alpha(A) < \infty$ and $q = \delta(A) < \infty$. If $p = q$, then clearly (5.11) holds true. Conversely, assume that $p \leq q$ and that (5.11) is satisfied. Since $q < \infty$, the identity (4.1) (with $i = q$ and $k = 1$) implies that $\text{dom } A^q = (\text{ran } A + \ker A^q) \cap \text{dom } A^q$, so that

$$\text{dom } A^q \subset \text{ran } A + \ker A^q. \quad (5.12)$$

It follows from (5.11) and (5.12) that $\text{dom } A^p \subset \text{ran } A + \ker A^q$. The inequality $p \leq q$ implies that $\ker A^p = \ker A^q$, and so

$$\text{dom } A^p \subset \text{ran } A + \ker A^p.$$

Then, the identity (4.1) (with $i = p$ and $k = 1$) implies that

$$\frac{\text{ran } A^p}{\text{ran } A^{p+1}} \cong \frac{\text{dom } A^p}{(\text{ran } A + \ker A^p) \cap \text{dom } A^p} = \{0\},$$

and hence $\text{ran } A^p = \text{ran } A^{p+1}$, which shows that $p \geq q$. Hence $p = q$. \square

Theorem 5.8. *Let A be a linear relation in a linear space \mathfrak{H} . Then:*

- (i) *If $\ker A^r \cap \text{ran } A^r = \{0\}$ and $\text{dom } A^r = (\text{dom } A^r \cap \text{ran } A^r) \oplus \ker A^r$ for some $r \in \mathbb{N}$, then $\alpha(A) \leq r$ and $\delta(A) \leq r$.*
- (ii) *Assume that $\mathfrak{R}_c(A) = \{0\}$. If $\alpha(A) < \infty$ and $q = \delta(A) < \infty$, then $\ker A^q \cap \text{ran } A^q = \{0\}$ and $\text{dom } A^q = (\text{dom } A^q \cap \text{ran } A^q) \oplus \ker A^q$.*

Proof. (i) Since $\ker A \subset \ker A^r$ it follows that $\ker A \cap \text{ran } A^r = \{0\}$. Then Lemma 5.5(i) implies that $\alpha(A) \leq r$. Next it will be shown that $\text{ran } A^r \subset \text{ran } A^{2r}$, so that $\text{ran } A^{2r} = \text{ran } A^r$, which implies $\delta(A) \leq r$. Let $y \in \text{ran } A^r$. Then there exists some $x \in \text{dom } A^r$ such that $\{x, y\} \in A^r$ and by hypothesis $x = x_1 + x_2$ with $x_1 \in \text{ran } A^r$ and $x_2 \in \ker A^r$. Now

$$\{x_1, y\} = \{x, y\} - \{x_2, 0\} \in A^r,$$

so that $y \in \text{ran } A^{2r}$. Hence $\text{ran } A^r \subset \text{ran } A^{2r}$.

(ii) Let $p = \alpha(A)$. Theorem 5.7 implies that $p \leq q$, and then Lemma 5.5(ii) leads to $\ker A^q \cap \text{ran } A^q = \{0\}$. Now the latter half of the statement will be proved. Clearly, $\text{dom } A^q \supset (\text{dom } A^q \cap \text{ran } A^q) \oplus \ker A^q$ holds true. As to the converse inclusion, consider the cases $q = 0$ and $q \geq 1$. If $q = 0$, then the desired inclusion is trivial. If $q \geq 1$, apply Lemma 5.6(ii) so that

$$\text{dom } A^q = (\text{dom } A^q \cap \text{ran } A^q) \oplus \mathfrak{M}_q,$$

which proves the desired inclusion since, by definition, $\mathfrak{M}_q \subset \ker A^q$. \square

5.3. Some remarks concerning a pair of relations

Assume now that A and B are relations in a linear space \mathfrak{H} such that $A \subset B$. It is clear that $\ker A \subset \ker B$ and $\text{ran } A \subset \text{ran } B$. Therefore

$$n(A) \leq n(B) \quad \text{and} \quad d(A) \geq d(B).$$

There is a similar inequality for the corresponding ascents.

Lemma 5.9. *Let A and B be relations in a linear space \mathfrak{H} such that $A \subset B$ and $\mathfrak{R}_c(B) = \{0\}$. Then $\alpha(A) \leq \alpha(B)$.*

Proof. The case $\alpha(B) = \infty$ is trivial, so assume that $\alpha(B) = p$ for some $p \in \mathbb{N} \cup \{0\}$. Let $x \in \ker A^{p+1}$, so that $\{x, y\} \in A^p$ and $\{y, 0\} \in A$ for some $y \in \mathfrak{H}$. Since $x \in \ker A^{p+1} \subset \ker B^{p+1} = \ker B^p$, it follows that $\{x, 0\} \in B^p$. Clearly, $\{x, y\} \in A^p \subset B^p$, so that also $\{0, y\} \in B^p$. Since $\{y, 0\} \in A \subset B$, the assumption $\mathfrak{R}_c(B) = \{0\}$ implies that $y = 0$, so that $x \in \ker A^p$. Hence $\ker A^{p+1} \subset \ker A^p$, so that $\alpha(A) \leq p$. \square

For linear relations $A \subset B$ the corresponding inequality for the descents (i.e. $\delta(A) \leq \delta(B)$) is not necessarily satisfied, even if $\mathfrak{R}_c(B)$ is trivial. Furthermore, it should be observed that the

condition $A \subset B$ in Lemma 5.9 cannot be replaced by the condition $\ker A^n \subset \ker B^n$ for all $n \in \mathbb{N} \cup \{0\}$, see Section 10.

6. Relating nullity and defect to ascent and descent

Let A be a relation in a linear space \mathfrak{S} . In this section the interrelations between nullity $n(A)$ and defect $d(A)$, and ascent $\alpha(A)$ and descent $\delta(A)$ are studied. In the main results the absence of singular chains for A has to be assumed.

6.1. Some preliminary observations

First some elementary relations between nullity and ascent, and defect and descent, respectively, are presented.

Lemma 6.1. *Let A be a relation in a linear space \mathfrak{S} . Assume there exists some $M \in \mathbb{N} \cup \{0\}$ such that $n(A^k) \leq M$ for $k \in \mathbb{N} \cup \{0\}$. Then $\alpha(A) \leq M$.*

Proof. If $\alpha(A) = \infty$ then $\ker A^{k+1} \supsetneq \ker A^k$, $k \in \mathbb{N} \cup \{0\}$. Hence $n(A^k) < n(A^{k+1})$, $k \in \mathbb{N} \cup \{0\}$, which implies that the sequence $n(A^k)$ is unbounded. This contradiction implies that $\alpha(A) < \infty$. Assume that $\alpha(A) = p$ for some $p \in \mathbb{N} \cup \{0\}$. In the case $p = 0$ the statement is trivial, so what remains to be shown is that $p \leq M$ if $p > 0$. Clearly,

$$\{0\} = \ker A^0 \subsetneq \ker A \subsetneq \cdots \subsetneq \ker A^{p-1} \subsetneq \ker A^p$$

and thus

$$0 = n(A^0) < n(A) < \cdots < n(A^{p-1}) < n(A^p).$$

Therefore, $p - 1 < n(A^p) \leq M$, leading to $p \leq M$. \square

Lemma 6.2. *Let A be a relation in a linear space \mathfrak{S} . Assume that there is some $M \in \mathbb{N} \cup \{0\}$ such that $d(A^k) \leq M$ for $k \in \mathbb{N} \cup \{0\}$. Then $\delta(A) \leq M$.*

Proof. If $\delta(A) = \infty$ then $\text{ran } A^{k+1} \subsetneq \text{ran } A^k$, $k \in \mathbb{N} \cup \{0\}$. Hence $d(A^{k+1}) > d(A^k)$, $k \in \mathbb{N} \cup \{0\}$, which implies that the sequence $d(A^k)$ is unbounded. This contradiction implies that $\delta(A) < \infty$. Assume that $\delta(A) = q$ for some $q \in \mathbb{N} \cup \{0\}$. The case $q = 0$ is obvious, so let $q > 0$. Since $\dim(\text{ran } A^k / \text{ran } A^{k+1}) > 0$ for $k < q$, it follows from (3.6) and Lemma 2.1 that

$$0 = d(A^0) < d(A) < \cdots < d(A^{q-1}) < d(A^q).$$

Therefore, $q - 1 < d(A^q) \leq M$, leading to $q \leq M$. \square

Recall that for a relation A one has $\alpha(A) = 0$ if and only if $n(A) = 0$. Therefore the product $\alpha(A)n(A)$ is well defined even when one of the factors is equal to ∞ .

Corollary 6.3. *Let A be a relation in a linear space \mathfrak{S} and let $k \in \mathbb{N} \cup \{0\}$. Then:*

- (i) $n(A^k) \leq \alpha(A)n(A)$;
- (ii) $d(A^k) \leq \delta(A)d(A)$.

Proof. (i) Clearly, it suffices to consider the case where both $\alpha(A)$ and $n(A)$ are finite. Let $\alpha(A) = p$, so that $n(A^k) \leq n(A^p)$. Furthermore, Lemma 5.4 implies that $n(A^p) \leq pn(A)$, which leads to

$$n(A^k) \leq n(A^p) \leq pn(A) = \alpha(A)n(A).$$

The statement in (ii) follows using similar arguments. \square

Theorem 6.4. Let A be a relation in a linear space \mathfrak{H} .

- (i) Assume that $\mathfrak{R}_c(A) = \{0\}$. If $\alpha(A) < \infty$, then $\alpha(A_{\mathfrak{M}}) = 0$ for every subspace \mathfrak{M} of \mathfrak{H} which is exactly range invariant under A .
- (ii) If $n(A) < \infty$ and $\alpha(A_{\mathfrak{M}}) = 0$ for every subspace \mathfrak{M} of \mathfrak{H} which is exactly range invariant under A , then $\alpha(A) < \infty$.

Proof. (i) Assume that $\mathfrak{R}_c(A) = \{0\}$ and $\alpha(A) < \infty$. Let $\mathfrak{M} \subset \mathfrak{H}$ be a subset which is exactly range invariant under A . Clearly $A_{\mathfrak{M}} \subset A$ and $\text{ran } A_{\mathfrak{M}} = \mathfrak{M}$ by hypothesis, so that $\delta(A_{\mathfrak{M}}) = 0$. According to Lemma 5.9 $\alpha(A_{\mathfrak{M}}) \leq \alpha(A) < \infty$. Hence Lemma 3.1 and Theorem 5.7 now imply $\alpha(A_{\mathfrak{M}}) \leq \delta(A_{\mathfrak{M}}) = 0$.

(ii) Conversely, assume that $n(A) < \infty$ and that $\alpha(A_{\mathfrak{M}}) = 0$ for any subspace $\mathfrak{M} \subset \mathfrak{H}$ which is exactly range invariant subspace under A , i.e., $\text{ran } A_{\mathfrak{M}} = \mathfrak{M}$. Consider the sequence of subspaces $\{\ker A \cap \text{ran } A^n\}_{n \in \mathbb{N}}$. Note that $\text{ran } A^{n+1} \subset \text{ran } A^n$ implies that $\ker A \cap \text{ran } A^{n+1} \subset \ker A \cap \text{ran } A^n$, so that

$$0 \leq \dots \leq \dim(\ker A \cap \text{ran } A^{n+1}) \leq \dim(\ker A \cap \text{ran } A^n) \leq \dots \leq n(A) < \infty.$$

Therefore, there exists some $r \in \mathbb{N} \cup \{0\}$ such that $\ker A \cap \text{ran } A^r = \ker A \cap \text{ran } A^n$ if $n \geq r$. Define the subspace \mathfrak{M} by

$$\mathfrak{M} = \bigcap_{i=0}^{\infty} \text{ran } A^{r+i}.$$

Observe that $\mathfrak{M} = \bigcap_{i=0}^{\infty} \text{ran } A^{r+i} = \bigcap_{i=j}^{\infty} \text{ran } A^{r+i}$ for any $j \in \mathbb{N} \cup \{0\}$ and that

$$\ker A \cap \mathfrak{M} = \ker A \cap \text{ran } A^r.$$

In order to show that \mathfrak{M} is exactly range invariant under A , it suffices to show that $\mathfrak{M} \subset \text{ran } A_{\mathfrak{M}}$. Let $y \in \mathfrak{M}$, then there exists a sequence $\{x_i\}_{i \geq 1}$ of elements in \mathfrak{H} such that $\{x_i, y\} \in A^{r+i}$. Since $A^{r+i} = AA^{r+i-1}$ for every x_i there is an element $x'_i \in \mathfrak{H}$ such that

$$\{x_i, x'_i\} \in A^{r+i-1} \quad \text{and} \quad \{x'_i, y\} \in A.$$

Let $u_i = x'_1 - x'_i$, so that $\{u_i, 0\} = \{x'_1, y\} - \{x'_i, y\} \in A$. Hence $u_i \in \ker A$. Now $x'_1 \in \text{ran } A^r$ and $x'_i \in \text{ran } A^{r+i-1} \subset \text{ran } A^r$, so that $u_i \in \ker A \cap \text{ran } A^r = \ker A \cap \text{ran } A^{r+i-1}$. But then also

$$x'_1 = u_i + x'_i \in \text{ran } A^{r+i-1} \quad \text{for all } i \in \mathbb{N},$$

so that $x'_1 \in \mathfrak{M}$, i.e. $\{x'_1, y\} \in A_{\mathfrak{M}}$ and $y \in \text{ran } A_{\mathfrak{M}}$. Hence \mathfrak{M} is exactly range invariant under A . Now, by hypothesis $\alpha(A_{\mathfrak{M}}) = 0$, i.e., $\ker A_{\mathfrak{M}} = \{0\}$. Therefore $\ker A \cap \text{ran } A^r = \ker A \cap \mathfrak{M} = \ker A_{\mathfrak{M}} = \{0\}$, which implies that $\alpha(A) \leq r$ by Lemma 5.5(i). \square

6.2. Relations with finite ascent or descent

If either the ascent or the descent is finite, it is possible to obtain inequalities involving the nullity and the defect.

Theorem 6.5. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. If $p = \alpha(A) < \infty$, then*

$$n(A) \leq d(A), \quad (6.1)$$

and there is equality in (6.1) if

$$\mathfrak{H} = \text{ran } A + \ker A^p. \quad (6.2)$$

Furthermore, if $p = \alpha(A) < \infty$ and $n(A) = d(A) < \infty$, then (6.2) holds.

Proof. Since $\mathfrak{R}_c(A) = \{0\}$ and $p < \infty$, it follows from Lemma 5.5(ii) that $\ker A \cap \text{ran } A^p = \{0\}$. Therefore, the results of this theorem now follow from Lemma 5.3. \square

The following result goes back to [12] for the case of linear operators. It remains valid in the context of relations.

Theorem 6.6. *Let A be a relation in a linear space \mathfrak{H} . If $q = \delta(A) < \infty$, then*

$$d(A) \leq n(A) + \dim \frac{\mathfrak{H}}{\text{dom } A^q + \text{ran } A}, \quad (6.3)$$

and there is equality in (6.3) if

$$\ker A \cap \text{ran } A^q = \{0\}. \quad (6.4)$$

Furthermore, if $q = \delta(A) < \infty$, $d(A) < \infty$, and there is equality in (6.3), then (6.4) holds.

Proof. As $q = \delta(A) < \infty$, the following inclusions are obvious:

$$\text{ran } A \subset \text{ran } A + \ker A^q \subset \text{ran } A + \text{dom } A^q \subset \mathfrak{H},$$

and a repeated application of Lemma 2.1 then gives

$$d(A) = \dim \frac{\mathfrak{H}}{\text{ran } A + \text{dom } A^q} + \dim \frac{\text{ran } A + \text{dom } A^q}{\text{ran } A + \ker A^q} + \dim \frac{\text{ran } A + \ker A^q}{\text{ran } A}. \quad (6.5)$$

It follows from (4.1) (with $i = q$ and $k = 1$) that

$$\text{dom } A^q \subset \text{ran } A + \ker A^q,$$

which implies that

$$\dim \frac{\text{ran } A + \text{dom } A^q}{\text{ran } A + \ker A^q} = 0. \quad (6.6)$$

Furthermore, Lemmas 2.3 and 4.3 lead to

$$\dim \frac{\text{ran } A + \ker A^q}{\text{ran } A} = \dim \frac{\ker A^q}{\text{ran } A \cap \ker A^q} = \dim \frac{\ker A}{\text{ran } A^q \cap \ker A}. \quad (6.7)$$

A combination of (6.5)–(6.7) gives

$$d(A) = \dim \frac{\mathfrak{H}}{\text{ran } A + \text{dom } A^q} + \dim \frac{\ker A}{\text{ran } A^q \cap \ker A}. \quad (6.8)$$

Clearly, $\dim(\ker A)/(\text{ran } A^q \cap \ker A) \leq n(A)$, so that (6.8) implies (6.4).

If, additionally, (6.4) holds, then clearly

$$n(A) = \dim \frac{\ker A}{\text{ran } A^q \cap \ker A},$$

and (6.8) implies equality in (6.3).

Assume now that $q = \delta(A) < \infty$, $d(A) < \infty$, and that there is equality in (6.3), so that

$$d(A) = n(A) + \dim \frac{\mathfrak{H}}{\text{dom } A^q + \text{ran } A} < \infty.$$

Then it follows from (6.8) that

$$n(A) = \dim \frac{\ker A}{\text{ran } A^q \cap \ker A},$$

and hence $\text{ran } A^q \cap \ker A = \{0\}$. \square

In particular, [21, Theorem 4.3] can be stated for relations as follows.

Corollary 6.7. *Let A be a relation in a linear space \mathfrak{H} . If $q = \delta(A) < \infty$, then*

$$d(A) \leq n(A) + \dim \frac{\mathfrak{H}}{\text{dom } A^q}. \quad (6.9)$$

If, in addition, $\text{dom } A = \mathfrak{H}$, then $d(A) \leq n(A)$.

Proof. Clearly, $\text{dom } A^q \subset \text{dom } A^q + \text{ran } A$, so that

$$\dim \frac{\mathfrak{H}}{\text{dom } A^q + \text{ran } A} \leq \dim \frac{\mathfrak{H}}{\text{dom } A^q}. \quad (6.10)$$

A combination of (6.3) and (6.10) leads to (6.9). If in addition $\text{dom } A = \mathfrak{H}$, then (6.9) and Corollary 3.6 imply that $d(A) \leq n(A)$. \square

A combination of Theorem 6.5 and Corollary 6.7 leads to the following result.

Corollary 6.8. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. Assume that $\alpha(A) < \infty$ and $q = \delta(A) < \infty$. Then*

$$n(A) \leq d(A) \leq n(A) + \dim \frac{\mathfrak{H}}{\text{dom } A^q + \text{ran } A}.$$

If, in addition, $\mathfrak{H} = \text{dom } A^q + \text{ran } A$, then $n(A) = d(A)$.

Theorem 6.9. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. Assume that $p = \alpha(A) < \infty$, $n(A) < \infty$, and*

$$\dim \frac{\text{dom } A^p}{\text{ran } A \cap \text{dom } A^p} \leq n(A). \quad (6.11)$$

Then $\alpha(A) = \delta(A)$ and there is actually equality in (6.11). If, in addition,

$$\text{ran } A + \text{dom } A^p = \mathfrak{H}, \quad (6.12)$$

then $n(A) = d(A)$.

Proof. It follows from (6.11), Lemmas 2.3, and 2.1 that

$$\begin{aligned} n(A) &\geq \dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p} = \dim \frac{\operatorname{dom} A^p + \operatorname{ran} A}{\operatorname{ran} A} \\ &= \dim \frac{\operatorname{dom} A^p + \operatorname{ran} A}{\ker A^p + \operatorname{ran} A} + \dim \frac{\ker A^p + \operatorname{ran} A}{\operatorname{ran} A}. \end{aligned} \quad (6.13)$$

Observe that

$$\frac{\operatorname{dom} A^p + \operatorname{ran} A}{\ker A^p + \operatorname{ran} A} \cong \frac{\operatorname{dom} A^p + (\ker A^p + \operatorname{ran} A)}{\ker A^p + \operatorname{ran} A}.$$

Hence, it follows from Lemma 2.3 and (4.1) (with $i = p$ and $k = 1$) that

$$\dim \frac{\operatorname{dom} A^p + \operatorname{ran} A}{\ker A^p + \operatorname{ran} A} = \frac{\dim \operatorname{dom} A^p}{(\ker A^p + \operatorname{ran} A) \cap \operatorname{dom} A^p} = \dim \frac{\operatorname{ran} A^p}{\operatorname{ran} A^{p+1}}. \quad (6.14)$$

Furthermore, it follows from Lemma 2.3 and (4.7) (with $i = p$) that

$$\dim \frac{\ker A^p + \operatorname{ran} A}{\operatorname{ran} A} = \dim \frac{\ker A^p}{\operatorname{ran} A \cap \ker A^p} = \dim \frac{\ker A}{\ker A \cap \operatorname{ran} A^p}.$$

Hence, (4.12) (with $k = 1$ and $i = p$) leads to

$$\begin{aligned} \dim \frac{\ker A^p + \operatorname{ran} A}{\operatorname{ran} A} &= n(A) - \dim(\ker A \cap \operatorname{ran} A^p) \\ &= n(A) - \dim \frac{\ker A^{p+1}}{\ker A^p} = n(A). \end{aligned} \quad (6.15)$$

A combination of (6.13)–(6.15) implies that

$$n(A) \geq \dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p} \geq \dim \frac{\operatorname{ran} A^p}{\operatorname{ran} A^{p+1}} + n(A) \geq n(A). \quad (6.16)$$

It follows from (6.16) that there is equality in (6.11). Furthermore it follows from (6.16) that $\dim \operatorname{ran} A^p / \operatorname{ran} A^{p+1} = 0$, which shows that $\delta(A) \leq \alpha(A)$. Thus, with $\alpha(A)$ also $\delta(A)$ is finite and Theorem 5.7 leads to $\alpha(A) \leq \delta(A)$, so that $\alpha(A) = \delta(A)$.

Finally, if (6.12) holds, then (6.11) and Lemma 5.2 (with $r = p$) show that $d(A) \leq n(A)$. By Theorem 6.5 it follows that $d(A) = n(A)$. This completes the proof. \square

6.3. Relations with finite ascent or descent and finite nullity or defect

Now a number of results are presented where the ascent or descent and the nullity or defect are assumed to be finite.

Theorem 6.10. *Let A be a relation in a linear space \mathfrak{H} . Assume that $q = \delta(A) < \infty$, $n(A) < \infty$, and*

$$n(A) \leq \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}. \quad (6.17)$$

Then $\alpha(A) \leq \delta(A)$, there is actually equality in (6.17), and

$$\dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q} = n(A). \quad (6.18)$$

If, in addition,

$$\text{ran } A + \text{dom } A^q = \mathfrak{H}, \quad (6.19)$$

then $n(A) = d(A)$.

Proof. By (4.7) (with $i = q$) and (6.17) it follows that

$$n(A) \geq \dim \frac{\ker A}{\ker A \cap \text{ran } A^q} = \dim \frac{\ker A^q}{\text{ran } A \cap \ker A^q} \geq n(A), \quad (6.20)$$

so that there is equality in (6.20). Hence, there is equality in (6.17) and

$$\ker A \cap \text{ran } A^q = \{0\}.$$

This last identity and Lemma 5.5(i) imply that $\alpha(A) \leq \delta(A)$. Furthermore, recall that (4.1) (with $i = q$ and $k = 1$) implies that

$$\frac{\text{dom } A^q}{(\text{ran } A + \ker A^q) \cap \text{dom } A^q} = \frac{\text{ran } A^q}{\text{ran } A^{q+1}} = \{0\},$$

so that $\text{dom } A^q \subset \text{ran } A + \ker A^q$, which shows that

$$\text{ran } A + \text{dom } A^q = \text{ran } A + \ker A^q. \quad (6.21)$$

It follows from (6.21) and repeated application of Lemma 2.3 that

$$\begin{aligned} \dim \frac{\text{dom } A^q}{\text{ran } A \cap \text{dom } A^q} &= \dim \frac{\text{dom } A^q + \text{ran } A}{\text{ran } A} \\ &= \dim \frac{\text{ran } A + \ker A^q}{\text{ran } A} = \dim \frac{\ker A^q}{\text{ran } A \cap \ker A^q}, \end{aligned}$$

so that the equality in (6.17) takes the form (6.18).

Finally, if (6.19) holds, then (6.18) and Lemma 5.2 (with $r = q$) show that $n(A) = d(A)$. This completes the proof. \square

Theorem 6.11. Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. Assume that $p = \alpha(A) < \infty$ and $n(A) = d(A) < \infty$. Then:

- (i) $\alpha(A) = \delta(A)$;
- (ii) $n(A^i) = d(A^i) < \infty$, $i \in \mathbb{N}$;
- (iii) $\mathfrak{H} = \text{ran } A^p \oplus \ker A^p$.

Proof. (i) It follows from Theorem 6.5 that $\mathfrak{H} = \text{ran } A + \ker A^p$. Then (4.1) (with $i = p$ and $k = 1$) leads to

$$\frac{\text{ran } A^p}{\text{ran } A^{p+1}} \cong \frac{\text{dom } A^p}{(\text{ran } A + \ker A^p) \cap \text{dom } A^p} = \{0\},$$

and hence $\delta(A) \leq \alpha(A) < \infty$. It is already known from Theorem 5.7 that $\alpha(A) \leq \delta(A)$, so that $\alpha(A) = \delta(A) < \infty$.

(ii) Let $k \in \mathbb{N} \cup \{0\}$. It follows from $\text{ran } A^{k+1} \subset \text{ran } A^k \subset \mathfrak{H}$ and Lemma 2.1 that

$$\dim \frac{\mathfrak{H}}{\text{ran } A^{k+1}} = \dim \frac{\mathfrak{H}}{\text{ran } A^k} + \dim \frac{\text{ran } A^k}{\text{ran } A^{k+1}},$$

or, in other words,

$$d(A^{k+1}) = d(A^k) + \dim \frac{\text{ran } A^k}{\text{ran } A^{k+1}}. \quad (6.22)$$

Since $d(A) < \infty$, it follows from Lemma 5.4(ii) that $d(A^k) \leq d(A^{k+1}) < \infty$, and then (6.22) and (4.1) lead to

$$d(A^{k+1}) - d(A^k) = \dim \frac{\text{dom } A^k}{(\text{ran } A + \ker A^k) \cap \text{dom } A^k}. \quad (6.23)$$

Using the identity

$$\text{ran } A + \ker A^k + \text{dom } A^k = \text{ran } A + \text{dom } A^k$$

and Lemma 2.3, the relation (6.23) can be written as

$$d(A^{k+1}) - d(A^k) = \dim \frac{\text{ran } A + \text{dom } A^k}{\text{ran } A + \ker A^k}. \quad (6.24)$$

Since $n(A) = d(A)$, and since $p = \alpha(A) < \infty$, it follows by Theorem 6.5 that

$$\mathfrak{H} = \ker A^p + \text{ran } A.$$

Using (3.8), the above identity shows that

$$\mathfrak{H} = \text{ran } A + \ker A^p \subset \text{ran } A + \text{dom } A^k \subset \mathfrak{H},$$

so that $\mathfrak{H} = \text{ran } A + \text{dom } A^k$ and then (6.24) implies that

$$d(A^{k+1}) - d(A^k) = \dim \frac{\mathfrak{H}}{\text{ran } A + \ker A^k}. \quad (6.25)$$

Since $\text{ran } A \subset \text{ran } A + \ker A^k \subset \mathfrak{H}$ it follows from Lemma 2.1 that

$$d(A) = \dim \frac{\mathfrak{H}}{\text{ran } A + \ker A^k} + \dim \frac{\text{ran } A + \ker A^k}{\text{ran } A} < \infty. \quad (6.26)$$

Using Lemma 2.3 and (4.7) (with $i = k$) it follows that

$$\dim \frac{\text{ran } A + \ker A^k}{\text{ran } A} = \dim \frac{\ker A^k}{\text{ran } A \cap \ker A^k} = \dim \frac{\ker A}{\ker A \cap \text{ran } A^k}. \quad (6.27)$$

The fact that $\dim \ker A = n(A) < \infty$ implies that

$$\dim \frac{\ker A}{\ker A \cap \text{ran } A^k} = n(A) - \dim(\ker A \cap \text{ran } A^k). \quad (6.28)$$

Combining (6.26)–(6.28), and using $n(A) = d(A)$ gives

$$\dim \frac{\mathfrak{H}}{\text{ran } A + \ker A^k} = \dim(\ker A \cap \text{ran } A^k). \quad (6.29)$$

Hence (6.25), (6.29), and (4.12) show that

$$d(A^{k+1}) - d(A^k) = \dim(\ker A \cap \text{ran } A^k) = \dim \frac{\ker A^{k+1}}{\ker A^k},$$

so that by Lemma 5.4

$$d(A^{k+1}) - d(A^k) = n(A^{k+1}) - n(A^k). \quad (6.30)$$

The relation (6.30) holds for all $k \in \mathbb{N}$. Hence $d(A^0) = n(A^0) = 0$ leads to

$$d(A^i) = \sum_{k=0}^{i-1} (d(A^{k+1}) - d(A^k)) = \sum_{k=0}^{i-1} (n(A^{k+1}) - n(A^k)) = n(A^i), \quad i \in \mathbb{N},$$

which completes the proof of (ii).

(iii) Since $p = \alpha(A) < \infty$, Lemma 4.4 shows that

$$\ker A^p \cap \operatorname{ran} A^p = \{0\}, \quad (6.31)$$

so that with Lemma 2.4 it follows that

$$n(A^p) = \dim \frac{\ker A^p}{\operatorname{ran} A^p \cap \ker A^p} \leq \dim \frac{\mathfrak{H}}{\operatorname{ran} A^p} = d(A^p). \quad (6.32)$$

Since $n(A^p) = d(A^p) < \infty$ by (ii), the inequality in (6.32) gives rise to

$$\dim \frac{\ker A^p}{\operatorname{ran} A^p \cap \ker A^p} = \dim \frac{\mathfrak{H}}{\operatorname{ran} A^p} < \infty.$$

Therefore Lemma 2.5 implies that $\mathfrak{H} = \operatorname{ran} A^p + \ker A^p$. This fact together with (6.31) proves (iii). \square

Note that $p = \alpha(A) = \delta(A) < \infty$ and $n(A) = d(A) < \infty$ do not necessarily imply that $\operatorname{dom} A^p = \operatorname{dom} A^{p+1}$, as can be seen in the next result.

Theorem 6.12. *Let A be a relation in a linear space \mathfrak{H} . Assume that $\alpha(A) = \delta(A) < \infty$ and $n(A) = d(A) < \infty$. Then*

$$\frac{\operatorname{dom} A^i}{\operatorname{dom} A^{i+1}} \cong \frac{\mathfrak{H}}{\operatorname{dom} A + \operatorname{mul} A^i}, \quad i \in \mathbb{N} \cup \{0\}. \quad (6.33)$$

Proof. Let $i \in \mathbb{N} \cup \{0\}$. Since $p = \alpha(A) = \delta(A) < \infty$ it follows that $\operatorname{ran} A^p \subset \operatorname{ran} A^i$. Furthermore, Theorem 6.11 implies that $\mathfrak{H} = \operatorname{ran} A^p \oplus \ker A^p$. Using the fact that $\ker A^p \subset \operatorname{dom} A$ it follows then that

$$\mathfrak{H} = \operatorname{ran} A^p + \ker A^p \subset \operatorname{ran} A^i + \ker A^p \subset \operatorname{ran} A^i + \operatorname{dom} A + \operatorname{mul} A^i \subset \mathfrak{H},$$

and hence

$$\mathfrak{H} = \operatorname{ran} A^i + \operatorname{dom} A + \operatorname{mul} A^i.$$

A repeated application of Lemma 2.3 implies that

$$\frac{\mathfrak{H}}{\operatorname{dom} A + \operatorname{mul} A^i} = \frac{\operatorname{dom} A + \operatorname{mul} A^i + \operatorname{ran} A^i}{\operatorname{dom} A + \operatorname{mul} A^i} \cong \frac{\operatorname{ran} A^i}{(\operatorname{dom} A + \operatorname{mul} A^i) \cap \operatorname{ran} A^i},$$

and then, using (4.2) it follows that (6.33) holds true. \square

Theorem 6.13. *Let A be a relation in a linear space \mathfrak{H} . Assume that $q = \delta(A) < \infty$ and $n(A) = d(A) < \infty$.*

- (i) *If $\mathfrak{H} = \operatorname{ran} A + \operatorname{dom} A^q$, then $\mathfrak{R}_c(A) = \{0\}$ and $\alpha(A) = \delta(A)$.*
- (ii) *Assume that $\mathfrak{R}_c(A) = \{0\}$. If $\alpha(A) = \delta(A)$, then $\mathfrak{H} = \operatorname{ran} A + \operatorname{dom} A^q$.*

Proof. (i) Assume that $\mathfrak{H} = \operatorname{ran} A + \operatorname{dom} A^q$ holds true, so that equality occurs in (6.3), and hence, since $d(A) < \infty$ and $\delta(A) < \infty$, it follows that $\ker A \cap \operatorname{ran} A^q = \{0\}$, which by Lemma 5.3

implies that $\mathfrak{R}_c(A) = \{0\}$. Then Lemma 5.5(i) shows that $p = \alpha(A) \leq q = \delta(A)$. Furthermore, since $\text{dom } A^p \subset \mathfrak{H} = \text{ran } A + \text{dom } A^q$, it follows from Theorem 5.7 that $\alpha(A) = \delta(A)$.

(ii) Assume that $\mathfrak{R}_c(A) = \{0\}$, that $n(A) = d(A) < \infty$, and that $q = \alpha(A) = \delta(A) < \infty$. Then Theorem 6.11(iii) shows that $\mathfrak{H} = \ker A^q \oplus \text{ran } A^q$ and since $\ker A^q \subset \text{dom } A^q$ and $\text{ran } A^q \subset \text{ran } A$, it follows that $\mathfrak{H} = \text{ran } A + \text{dom } A^q$. \square

7. Shifted linear relations

Let A be a linear relation in a linear space \mathfrak{H} . For any $\lambda \in \mathbb{K}$ the notation $A - \lambda$ stands for $A - \lambda I$, i.e.,

$$A - \lambda = \{(x, y - \lambda x) : (x, y) \in A\}.$$

It follows from the definition of the operator-like sum that

$$A - \lambda = (A - \mu) + (\mu - \lambda) \quad \text{for all } \lambda, \mu \in \mathbb{K}.$$

Observe that $\ker(A - \lambda) = \{x : (x, \lambda x) \in A\}$, i.e., $\ker(A - \lambda)$ is the eigenspace corresponding to the eigenvector $\lambda \in \mathbb{K}$. Though many of the previous results can be applied when the relation A is replaced by the relation $A - \lambda$, $\lambda \in \mathbb{K}$, it remains to show some specific results concerning shifted relations.

The *root manifold* $\mathfrak{R}_\lambda(A)$, $\lambda \in \mathbb{K}$, is defined by

$$\mathfrak{R}_\lambda(A) = \bigcup_{i=1}^{\infty} \ker(A - \lambda)^i.$$

Clearly the root manifolds $\mathfrak{R}_\lambda(A)$, $\lambda \in \mathbb{K}$, are subspaces of $\text{dom } A \subset \mathfrak{H}$. Observe that $\mathfrak{R}_0(A - \lambda) = \mathfrak{R}_\lambda(A)$. There is a similar observation for the multivalued parts. It is clear from the definition that $\text{mul}(A - \lambda) = \text{mul } A$ for all $\lambda \in \mathbb{K}$, and in fact for each $i \in \mathbb{N}$ one has

$$\text{mul}(A - \lambda)^i = \text{mul } A^i,$$

so that $\mathfrak{R}_\infty(A - \lambda) = \mathfrak{R}_\infty(A)$. Recall the definition (3.3) of the singular chain manifold of a relation A . It is now clear that the singular chain manifold of a relation $A - \lambda$, $\lambda \in \mathbb{K}$, is given by

$$\mathfrak{R}_c(A - \lambda) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\infty(A).$$

Lemma 7.1. *Let A be a relation in a linear space \mathfrak{H} and let $\lambda \in \mathbb{K}$. Then $\mathfrak{R}_c(A) = \{0\}$ if and only if $\mathfrak{R}_c(A - \lambda) = \{0\}$.*

Proof. The statement is trivial for $\lambda = 0$, so the case $\lambda \neq 0$ is considered. Assume that $\mathfrak{R}_c(A) = \{0\}$ and that $\mathfrak{R}_c(A - \lambda) \neq \{0\}$. Then there exist nonzero $x_i \in \mathfrak{H}$, $1 \leq i \leq p$, such that

$$\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_{p-1}, x_p\}, \{x_p, 0\} \in A - \lambda.$$

This means that

$$\begin{aligned} &\{0, x_1\}, \{x_1, x_2 + \lambda x_1\}, \{x_2, x_3 + \lambda x_2\}, \dots, \{x_{p-2}, x_{p-1} + \lambda x_{p-2}\}, \\ &\{x_{p-1}, x_p + \lambda x_{p-1}\}, \{x_p, x_{p+1} + \lambda x_p\} \in A, \end{aligned}$$

where $x_{p+1} = 0$. Define $z_{m,n} \in \mathbb{K}$ for $0 \leq n \leq m \leq p+1$ by

$$z_{m,n} = (-1)^{m+n} \lambda^{m-n} \binom{p-n}{m-n}.$$

Then

$$\begin{cases} z_{p+1,n} = 0, & 0 \leq n \leq p+1, \\ z_{m,m} = 1, & 0 \leq m \leq p, \\ z_{m,n} + \lambda z_{m,n+1} = z_{m+1,n+1}, & 0 \leq n \leq m \leq p, \end{cases} \quad (7.1)$$

and it follows that

$$z_{k,0}\{0, x_1\} + \sum_{i=1}^k z_{k,i}\{x_i, x_{i+1} + \lambda x_i\} \in A, \quad 1 \leq k \leq p+1,$$

so that

$$\left\{ \sum_{i=1}^k z_{k,i} x_i, \sum_{i=1}^k (z_{k,i-1} + \lambda z_{k,i}) x_i + z_{k,k} x_{k+1} \right\} \in A, \quad 1 \leq k \leq p+1, \quad (7.2)$$

where $x_{p+1} = x_{p+2} = 0$. Define $y_k = \sum_{i=1}^k z_{k,i} x_i$ for $1 \leq k \leq p+1$. Then one finds from (7.2) and (7.1) that

$$\{y_k, y_{k+1}\} \in A, \quad 1 \leq k \leq p,$$

where $y_1 = x_1$ and $y_{p+1} = 0$. Therefore,

$$\{0, y_1\}, \{y_1, y_2\}, \dots, \{y_p, 0\} \in A,$$

contradicting the assumption that $\mathfrak{R}_c(A) = \{0\}$, because $y_1 = x_1 \neq 0$.

The converse implication follows in the same way if A is replaced by $A + \lambda$. \square

The next result goes back to [21, Lemma 3.9] where it is proved for operators under the additional restriction that $n = \delta(A) < \infty$.

Lemma 7.2. *Let A be a relation in a linear space \mathfrak{H} and let $\lambda \in \mathbb{K} \setminus \{0\}$. Then*

$$\ker(A - \lambda)^k \subset \operatorname{ran} A^n \quad (7.3)$$

for all $k, n \in \mathbb{N} \cup \{0\}$.

Proof. If either $k = 0$ or $n = 0$, the desired inclusion is trivial. Now assume that $n \geq 1$ and $k \geq 1$. The proof will be given by induction on $n \in \mathbb{N}$.

First consider the case $n = 1$. If $k = 1$, $x_0 \in \ker(A - \lambda)$ implies that $\{x_0, 0\} \in A - \lambda$, so that $\{x_0, \lambda x_0\} \in A$ and therefore $x_0 \in \operatorname{ran} A$, as $\lambda \neq 0$. If, on the other hand, $k \geq 2$, it can also be shown that (7.3) holds for $n = 1$. Let $x_0 \in \ker(A - \lambda)^k$. Then there exist elements x_1, x_2, \dots, x_{k-1} such that

$$\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{k-2}, x_{k-1}\}, \{x_{k-1}, 0\} \in A - \lambda,$$

which means that

$$\{x_0, x_1 + \lambda x_0\}, \{x_1, x_2 + \lambda x_1\}, \dots, \{x_{k-2}, x_{k-1} + \lambda x_{k-2}\}, \{x_{k-1}, \lambda x_{k-1}\} \in A. \quad (7.4)$$

Define $x_k = 0$ and take a suitable linear combination of the pairs from (7.4),

$$\sum_{i=0}^{k-1} (-\lambda)^{k-i-1} \{x_i, x_{i+1} + \lambda x_i\} \in A,$$

so that

$$\left\{ \sum_{i=0}^{k-1} (-\lambda)^{k-1-i} x_i, -(-\lambda)^k x_0 \right\} \in A,$$

which shows that $x_0 \in \text{ran } A$ as $\lambda \neq 0$.

Now assume that (7.3) is satisfied for some $n \in \mathbb{N}$ and all $k \in \mathbb{N}$. Take $x_0 \in \ker(A - \lambda)^k$ and consider elements x_1, \dots, x_{k-1} as above such that the relation (7.4) is satisfied. With $x_k = 0$ it follows that

$$\sum_{j=0}^{i-1} \binom{i-1}{i-1-j} \lambda^{i-1-j} \{x_j, x_{j+1} + \lambda x_j\} \in A, \quad 1 \leq i \leq k. \quad (7.5)$$

Define $z_i = \sum_{j=0}^{i-1} \binom{i-1}{j} \lambda^j x_{i-1-j}$ for $1 \leq i \leq k+1$, then (7.5) is equivalent to

$$\{z_i, z_{i+1}\} \in A, \quad 1 \leq i \leq k. \quad (7.6)$$

Observe that $z_1 = x_0$, and that $x_0 \in \text{ran } A^n$ (by the induction hypothesis) implies that all $z_i \in \text{ran } A^n$. Define

$$c_i = (-1)^{k-i} \lambda^{k-i} \binom{k}{k-i}, \quad 1 \leq i \leq k,$$

then one finds by a straightforward calculation that

$$\sum_{i=1}^k c_i z_{i+1} = (-1)^{k+1} \lambda^k x_0 + x_k = (-1)^{k+1} \lambda^k x_0.$$

This fact and the relation (7.6) imply that

$$\left\{ \sum_{i=1}^k c_i z_i, (-1)^{k+1} \lambda^k x_0 \right\} = \sum_{i=1}^k c_i \{z_i, z_{i+1}\} \in A.$$

Since all $z_i \in \text{ran } A^n$ it follows that $(-1)^{k+1} \lambda^k x_0 \in \text{ran } A^{n+1}$, and hence $x_0 \in \text{ran } A^{n+1}$ as $\lambda \neq 0$. \square

Lemma 7.3. Let A be a relation in a linear space \mathfrak{H} . Assume that $\text{dom } A = \mathfrak{H}$ and let $\lambda \in \mathbb{K} \setminus \{0\}$. Let $\mathfrak{H}_1 = \text{ran } A$ and $A_1 = A_{\mathfrak{H}_1}$. Then:

- (i) $\alpha(A - \lambda) = \alpha(A_1 - \lambda)$ and $\text{n}(A - \lambda) = \text{n}(A_1 - \lambda)$.
- (ii) If \mathfrak{M} is a subspace of \mathfrak{H}_1 such that $\mathfrak{H}_1 = \text{ran}(A_1 - \lambda) \oplus \mathfrak{M}$, then also $\mathfrak{H} = \text{ran}(A - \lambda) \oplus \mathfrak{M}$. In particular, $\text{d}(A - \lambda) = \text{d}(A_1 - \lambda)$.
- (iii) Assume that $\mathfrak{R}_c(A) = \{0\}$. If $\text{n}(A_1 - \lambda) = \text{d}(A_1 - \lambda) < \infty$ and if either $\alpha(A_1 - \lambda) < \infty$ or $\delta(A_1 - \lambda) < \infty$, then $\alpha(A - \lambda) = \delta(A - \lambda) < \infty$.

Proof. (i) By induction it will be shown that $\ker(A_1 - \lambda)^m = \ker(A - \lambda)^m$ for all $m \in \mathbb{N} \cup \{0\}$. This identity implies the statements in (i). Clearly, the identity is true for $m = 0$, so assume $m \geq 1$.

First it is shown that $\ker(A_1 - \lambda)^m \subset \ker(A - \lambda)^m$. To see this let $x_1 \in \ker(A_1 - \lambda)^m$. Then there exists $x_i \in \mathfrak{H}$, $2 \leq i \leq m+1$, with $x_{m+1} = 0$, such that

$$\{x_i, x_{i+1}\} \in A_1 - \lambda, \quad 1 \leq i \leq m.$$

Since A_1 is a restriction of A it follows that

$$\{x_i, x_{i+1}\} \in A - \lambda, \quad 1 \leq i \leq m.$$

Therefore $x_1 \in \ker(A - \lambda)^m$. Hence $\ker(A_1 - \lambda)^m \subset \ker(A - \lambda)^m$.

Now the converse inclusion will be shown. Let $x_1 \in \ker(A - \lambda)^m$, so that there exist $x_2, x_3, \dots, x_m \in \mathfrak{H}$ such that

$$\{x_1, x_2 + \lambda x_1\}, \{x_2, x_3 + \lambda x_2\}, \dots, \{x_{m-1}, x_m + \lambda x_{m-1}\}, \{x_m, \lambda x_m\} \in A.$$

Clearly, $\lambda x_m \in \text{ran } A$ so that also $x_m \in \text{ran } A$, since $\lambda \neq 0$. Because of $x_m + \lambda x_{m-1} \in \text{ran } A$ it follows that $\lambda x_{m-1} \in \text{ran } A$. Inductively, it follows that all the elements x_m, \dots, x_1 are in $\text{ran } A = \mathfrak{H}_1$. Hence all pairs listed above are also in A_1 so that $x_1 \in \ker(A_1 - \lambda)^m$.

(ii) First it is shown that $\text{ran}(A_1 - \lambda) = \text{ran}(A - \lambda) \cap \mathfrak{H}_1$. The inclusion $\text{ran}(A_1 - \lambda) \subset \text{ran}(A - \lambda) \cap \mathfrak{H}_1$ is clear. To see the converse inclusion, let $y \in \text{ran}(A - \lambda) \cap \mathfrak{H}_1$. Then for some $x \in \mathfrak{H}$, $\{x, y + \lambda x\} \in A$, i.e., $y + \lambda x \in \text{ran } A = \mathfrak{H}_1$. But then $\lambda x \in \mathfrak{H}_1$ so that $\{x, y + \lambda x\} \in A_1$ and therefore $y \in \text{ran}(A_1 - \lambda)$. Hence $\text{ran}(A - \lambda) \cap \mathfrak{H}_1 \subset \text{ran}(A_1 - \lambda)$.

Now let \mathfrak{M} be a subspace of \mathfrak{H}_1 which is complementary to $\text{ran}(A_1 - \lambda)$, i.e., $\mathfrak{H}_1 = \text{ran}(A_1 - \lambda) \oplus \mathfrak{M}$, then it follows that

$$\mathfrak{M} \cap \text{ran}(A - \lambda) = \mathfrak{M} \cap \mathfrak{H}_1 \cap \text{ran}(A - \lambda) = \mathfrak{M} \cap \text{ran}(A_1 - \lambda) = \{0\}.$$

Clearly $\text{ran}(A - \lambda) + \mathfrak{M} \subset \mathfrak{H}$. In order to show the converse inclusion, let $x \in \mathfrak{H}$. Because of $\text{dom}(\lambda - A) = \text{dom } A = \mathfrak{H}$ there is some $y \in \text{ran}(A - \lambda)$ such that $\{x, y\} \in A - \lambda$, i.e. $\{x, y + \lambda x\} \in A$. Then $y + \lambda x \in \mathfrak{H}_1$ and by the hypothesis there exist $u \in \text{ran}(A_1 - \lambda) \subset \text{ran}(A - \lambda)$ and $v \in \mathfrak{M}$ such that $y + \lambda x = u + v$. Now

$$x = \frac{u - y}{\lambda} + \frac{v}{\lambda} \in \text{ran}(A - \lambda) + \mathfrak{M},$$

which proves (ii).

(iii) It follows from the assumption $n(A_1 - \lambda) = d(A_1 - \lambda) < \infty$ and from (i) and (ii) that $n(A - \lambda) = d(A - \lambda) < \infty$. The condition $\mathfrak{R}_c(A) = \{0\}$ shows that $\mathfrak{R}_c(A - \lambda) = \{0\}$ and that $\mathfrak{R}_c(A_1 - \lambda) = \{0\}$ by Lemma 3.1 and Lemma 7.1.

If $\alpha(A_1 - \lambda) < \infty$, then Theorem 6.11 applied to $A_1 - \lambda$ leads to $\alpha(A_1 - \lambda) = \delta(A_1 - \lambda)$. Hence by (i) $\alpha(A - \lambda) = \alpha(A_1 - \lambda) < \infty$. It follows from Theorem 6.11 applied to $A - \lambda$ that $\alpha(A - \lambda) = \delta(A - \lambda)$.

If $\delta(A_1 - \lambda) < \infty$, Theorem 6.13 applied to $A_1 - \lambda$ shows that $\alpha(A_1 - \lambda) = \delta(A_1 - \lambda)$, since $\text{dom}(A_1 - \lambda) = \mathfrak{H}$. Hence by (i) $\alpha(A - \lambda) = \alpha(A_1 - \lambda) < \infty$, and it follows from Theorem 6.11 applied to $A - \lambda$ that $\alpha(A - \lambda) = \delta(A - \lambda)$. \square

Theorem 7.4. *Let A be a relation in a linear space \mathfrak{H} with $\mathfrak{R}_c(A) = \{0\}$. Assume that $\text{dom } A = \mathfrak{H}$ and $\dim \text{ran } A < \infty$. Then $n(A - \lambda) = d(A - \lambda) < \infty$ and $\alpha(A - \lambda) = \delta(A - \lambda) < \infty$ for all $\lambda \in \mathbb{K} \setminus \{0\}$.*

Proof. Let A_1 and \mathfrak{H}_1 be as in Lemma 7.3. Then $\dim \mathfrak{H}_1 < \infty$, and it follows that $n(A_1 - \lambda)$, $d(A_1 - \lambda)$, $\alpha(A_1 - \lambda)$, and $\delta(A_1 - \lambda)$ are all finite. Then, by Corollary 6.8, $n(A_1 - \lambda) = d(A_1 - \lambda)$. Finally, it follows from Lemma 7.3 that $n(A - \lambda) = d(A - \lambda) < \infty$ and $\alpha(A - \lambda) = \delta(A - \lambda) < \infty$. \square

8. Completely reduced relations

Let A be a relation in a linear space \mathfrak{H} and assume that \mathfrak{M}_1 and \mathfrak{M}_2 are two complementary subspaces of \mathfrak{H} . Observe that

$$A_1 \hat{+} A_2 \subset A,$$

where $A_i = A|_{\mathfrak{M}_i}$, i.e., $A_i = A \cap (\mathfrak{M}_i \times \mathfrak{M}_i)$, $i = 1, 2$. The relation A is said to be *completely reduced* by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$ if it can be decomposed as

$$A = A_1 \oplus A_2, \quad (8.1)$$

where the notation indicates the component-wise direct sum decomposition $A = A_1 \hat{+} A_2$, so that also $A_1 \cap A_2 = \{\{0, 0\}\}$. In the case of linear operators the concept of complete reducibility goes back at least to Taylor (see [21, Section 6]) whose definition is slightly different but equivalent with (8.1). If the relation A is completely reduced by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$, then

$$\text{dom } A = \text{dom } A_1 \oplus \text{dom } A_2, \quad \ker A = \ker A_1 \oplus \ker A_2, \quad (8.2)$$

and

$$\text{ran } A = \text{ran } A_1 \oplus \text{ran } A_2, \quad \text{mul } A = \text{mul } A_1 \oplus \text{mul } A_2. \quad (8.3)$$

Furthermore, note that $\text{dom } A = \mathfrak{H}$ or $\text{ran } A = \mathfrak{H}$ if and only if $\text{dom } A_i = \mathfrak{M}_i$ or $\text{ran } A_i = \mathfrak{M}_i$, $i = 1, 2$, respectively.

Lemma 8.1. *Let A be a relation in a linear space \mathfrak{H} and let \mathfrak{M}_1 and \mathfrak{M}_2 be two complementary subspaces of \mathfrak{H} . Assume that A is completely reduced by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$ and let $n \in \mathbb{N}$. Then A^n is completely reduced by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$ and*

$$A^n = A_1^n \oplus A_2^n. \quad (8.4)$$

Proof. Clearly $A_i^n \subset A^n$ and $A_i^n \subset \mathfrak{M}_i \times \mathfrak{M}_i$, $1 \leq i \leq 2$, so that $A_1^n \cap A_2^n = \{\{0, 0\}\}$. Thus for all $n \in \mathbb{N}$

$$A_1^n \hat{+} A_2^n \subset A^n, \quad \text{direct sum.} \quad (8.5)$$

Now for all $n \in \mathbb{N}$ the converse inclusion

$$A^n \subset A_1^n \hat{+} A_2^n \quad (8.6)$$

will be shown. The inclusions (8.5) and (8.6) imply the identity (8.4).

By hypothesis the inclusion in (8.6) holds true for $n = 1$ and assume that (8.6) holds true for $n = k \in \mathbb{N}$. Let $\{x, y\} \in A^{k+1} = (A_1 \oplus A_2)^{k+1}$, so that

$$\{x, z\} \in (A_1 \oplus A_2)^k \quad \text{and} \quad \{z, y\} \in A_1 \oplus A_2,$$

for some $z \in \mathfrak{H}$. The induction assumption implies

$$\{x, z\} \in (A_1 \oplus A_2)^k \subset A_1^k \oplus A_2^k.$$

Therefore there exist $\{x_i, z_i\} \in A_i^k$, $\{z'_i, y_i\} \in A_i$, $1 \leq i \leq 2$, such that

$$\{x, z\} = \{x_1, z_1\} + \{x_2, z_2\}, \quad \{z, y\} = \{z'_1, y_1\} + \{z'_2, y_2\}.$$

Clearly, it follows that

$$x = x_1 + x_2, \quad y = y_1 + y_2, \quad z = z_1 + z_2 = z'_1 + z'_2.$$

In particular,

$$z_1 - z'_1 = z'_2 - z_2 \in \mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}.$$

Therefore $z_i = z'_i$ and thus $\{x_i, y_i\} \in A_i^{k+1}$, $1 \leq i \leq 2$. It follows that $\{x, y\} \in A_1^{k+1} \hat{+} A_2^{k+1}$. Hence (8.6) is valid for all $n \in \mathbb{N}$. This completes the proof. \square

Theorem 8.2. *Let A be a relation in a linear space \mathfrak{H} and let \mathfrak{M}_1 and \mathfrak{M}_2 be two complementary subspaces of \mathfrak{H} . Assume that A is completely reduced by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$. Then:*

- (i) $n(A) < \infty$ if and only if $n(A_1) < \infty$ and $n(A_2) < \infty$, and in this case $n(A) = n(A_1) + n(A_2)$.
- (ii) $d(A) < \infty$ if and only if $d(A_1) < \infty$ and $d(A_2) < \infty$, and in this case $d(A) = d(A_1) + d(A_2)$.
- (iii) If $\alpha(A) = p < \infty$, then $\alpha(A_i) \leq p$, $i = 1, 2$. If $\alpha(A_i) = p_i < \infty$, $i = 1, 2$, then $\alpha(A) = \max(p_1, p_2)$.
- (iv) If $\delta(A) = q < \infty$, then $\delta(A_i) \leq q$, $i = 1, 2$. If $\delta(A_i) = q_i < \infty$, $i = 1, 2$, then $\delta(A) = \max(q_1, q_2)$.
- (v) If $n(A) = d(A) < \infty$, $q = \delta(A) < \infty$, and $\mathfrak{H} = \text{ran } A + \text{dom } A^q$, then $n(A_i) = d(A_i) < \infty$ and $\alpha(A_i) = \delta(A_i) < \infty$, $i = 1, 2$.
In addition, assume that $\mathfrak{R}_c(A) = \{0\}$. Then:
 - (vi) If $n(A) = d(A) < \infty$ and $\alpha(A) < \infty$, then $n(A_i) = d(A_i) < \infty$ and $\alpha(A_i) = \delta(A_i) < \infty$, $i = 1, 2$.
 - (vii) If $n(A_i) = d(A_i) < \infty$ and $\alpha(A_i) < \infty$, $i = 1, 2$, then $n(A) = d(A) < \infty$ and $\alpha(A) = \delta(A) < \infty$, $i = 1, 2$.

Proof. (i) This statement follows from the latter identity in (8.2).

(ii) Let \mathfrak{U} be a complement of $\text{ran } A$ in \mathfrak{H} , and let \mathfrak{U}_i be a complement of $\text{ran } A_i$ in \mathfrak{M}_i , $i = 1, 2$. The first identity in (8.3) leads to

$$\text{ran } A_1 \oplus \text{ran } A_2 \oplus \mathfrak{U} = \text{ran } A_1 \oplus \text{ran } A_2 \oplus \mathfrak{U}_1 \oplus \mathfrak{U}_2,$$

which shows that $\mathfrak{U}_1 \oplus \mathfrak{U}_2$ is also a complement of $\text{ran } A$ in \mathfrak{H} , so that (ii) follows.

(iii) Lemma 8.1 shows that A^n is completely reduced by the pair $(\mathfrak{M}_1, \mathfrak{M}_2)$ and hence (8.2) and (8.3) can be applied with A^n instead of A .

Assume now that $\alpha(A) = p < \infty$. Let $x \in \ker A_1^{p+1}$, so that

$$x \in \ker A^{p+1} = \ker A^p = \ker A_1^p \oplus \ker A_2^p.$$

Then $x = x_1 + x_2$ for some $x_i \in \ker A_i^p$, $i = 1, 2$, so that $x_2 = x - x_1 \in \ker A_1^{p+1}$ and $x_2 \in \ker A_2^p \subset \ker A_2^{p+1}$, which shows that $x_2 = x - x_1 = 0$, and so $x = x_1 \in \ker A_1^p$. Thus $\alpha(A_1) \leq p$ and by symmetry it follows that $\alpha(A_2) \leq p$.

Conversely, assume that $\alpha(A_i) = p_i < \infty$, $i = 1, 2$, and let $p = \max(p_1, p_2)$, so that

$$\ker A^{p+1} = \ker A_1^{p+1} \oplus \ker A_2^{p+1} = \ker A_1^p \oplus \ker A_2^p = \ker A^p,$$

which implies that $\alpha(A) \leq p = \max(p_1, p_2)$. Furthermore, it follows from the first part of the proof that $\max(p_1, p_2) \leq \alpha(A)$. Thus $\alpha(A) = \max(p_1, p_2)$.

(iv) The proof is similar to that of (iii).

(v) Theorem 6.13 implies that $\alpha(A) = \delta(A)$ and that $\mathfrak{R}_c(A) = \{0\}$. By (i)–(iv) it follows that $n(A_i)$, $d(A_i)$, $\alpha(A_i)$ and $\delta(A_i)$, $i = 1, 2$, are all finite. Furthermore, Theorem 6.5 shows that $n(A_i) \leq d(A_i)$, $i = 1, 2$. Clearly,

$$n(A_1) + n(A_2) = n(A) = d(A) = d(A_1) + d(A_2),$$

so that

$$n(A_1) - d(A_1) = d(A_2) - n(A_2).$$

Since in the above equality the left-hand side is nonpositive and the right-hand side is nonnegative it follows that both $n(A_1) - d(A_1)$ and $d(A_2) - n(A_2)$ are equal to zero. Apply again Theorem 6.11 to conclude that $\alpha(A_i) = \delta(A_i)$, $i = 1, 2$.

(vi) Theorem 6.11 implies that $\alpha(A) = \delta(A)$ and the proof proceeds as in (v).

(vii) Theorem 6.11 implies that $\alpha(A_i) = \delta(A_i) < \infty$, $i = 1, 2$. The statement follows now by applying (i)–(iv). \square

Theorem 8.3. *Let A be a relation in a linear space \mathfrak{H} . Then:*

(i) *If $\mathfrak{H} = \text{ran } A^p \oplus \ker A^p$ for some $p \in \mathbb{N} \cup \{0\}$, then $\alpha(A) \leq p$, $\delta(A) \leq p$, and A is completely reduced by the pair $(\text{ran } A^p, \ker A^p)$.*

In addition, assume that $\mathfrak{R}_c(A) = \{0\}$.

(ii) *If $p = \alpha(A) < \infty$ and $n(A) = d(A) < \infty$, then A is completely reduced by the pair $(\text{ran } A^p, \ker A^p)$.*

(iii) *If $\text{dom } A = \mathfrak{H}$, $\alpha(A) < \infty$, and $q = \delta(A) < \infty$, then A is completely reduced by the pair $(\text{ran } A^q, \ker A^q)$.*

Proof. (i) Assume that $\text{ran } A^p$ and $\ker A^p$ are complementary subspaces. In order to show $\alpha(A) \leq p$, let $x \in \ker A^{p+1}$, so that $\{x, 0\} \in A^{p+1}$. Then $\{x, y\} \in A^p$ and $\{y, 0\} \in A$ for some $y \in \mathfrak{H}$. Hence $y \in \text{ran } A^p$ but also $y \in \ker A \subset \ker A^p$, which implies that $y = 0$, since $\text{ran } A^p \cap \ker A^p = \{0\}$. Hence $x \in \ker A^p$. Therefore $\ker A^{p+1} = \ker A^p$ and, thus, $\alpha(A) \leq p$. In order to show $\delta(A) \leq p$, let $y \in \text{ran } A^p$. Then $\{x, y\} \in A^p$ for some $x \in \mathfrak{H}$. Then $x = x_1 + x_2$ with $x_1 \in \text{ran } A^p$ and $x_2 \in \ker A^p$, since $\mathfrak{H} = \text{ran } A^p + \ker A^p$. Hence $\{x_2, 0\} \in A^p$ and $\{u, x_1\} \in A^p$ for some $u \in \mathfrak{H}$. Then

$$\{x_1, y\} = \{x, y\} - \{x_2, 0\} \in A^p,$$

which shows that $\{u, y\} \in A^{2p}$, i.e., $y \in \text{ran } A^{2p}$. Therefore $\text{ran } A^p = \text{ran } A^{2p}$ and, thus $\delta(A) \leq p$.

Let $\mathfrak{M} = \text{ran } A^p$ and $\mathfrak{N} = \ker A^p$. In order to prove that the pair $(\mathfrak{M}, \mathfrak{N})$ completely reduces A it suffices to show that $A \subset A_{\mathfrak{M}} \oplus A_{\mathfrak{N}}$. Let $\{x, y\} \in A$, so that $x = x_1 + x_2$ for some $x_1 \in \text{ran } A^p$ and $x_2 \in \ker A^p$. Then $\{x_2, y_2\} \in A$ and $\{y_2, 0\} \in A^{p-1}$ for some $y_2 \in \mathfrak{H}$. Clearly,

$$\{x_1, y - y_2\} = \{x, y\} - \{x_2, y_2\} \in A.$$

Since $x_1 \in \text{ran } A^p$ and $\{x_1, y - y_2\} \in A$, it follows that $y - y_2 \in \text{ran } A^{p+1} = \text{ran } A^p = \mathfrak{M}$, so that $\{x_1, y - y_2\} \in A_{\mathfrak{M}}$. Furthermore $y_2 \in \ker A^{p-1} \subset \ker A^p$ and $x_2 \in \ker A^p$ imply that $\{x_2, y_2\} \in A_{\mathfrak{N}}$, so that

$$\{x, y\} = \{x_1, y - y_2\} + \{x_2, y_2\} \in A_{\mathfrak{M}} \oplus A_{\mathfrak{N}}.$$

Hence $A \subset A_{\mathfrak{M}} \oplus A_{\mathfrak{N}}$, and the proof is complete.

(ii) It follows from Theorem 6.11 that $p = \alpha(A) = \delta(A)$ and, furthermore, that $\mathfrak{H} = \text{ran } A^p \oplus \ker A^p$. Thus (i) implies (ii).

(iii) It follows from Theorem 5.8 that $\mathfrak{H} = \text{ran } A^q \oplus \ker A^q$, so that again (i) implies (ii). \square

9. Decomposition results

This section is concerned with the decomposition of a relation A in a linear space \mathfrak{H} as an operator-like sum $A = A_1 + B$, where A_1 is a relation in \mathfrak{H} with nice properties such as $n(A_1) = d(A_1) = 0$, and B is an everywhere defined operator in \mathfrak{H} with $\dim \operatorname{ran} B < \infty$. All considerations are entirely algebraic and the assumption $\operatorname{dom} A = \mathfrak{H}$ is not assumed except where explicitly stated.

Theorem 9.1. *Let A be a relation in a linear space \mathfrak{H} . Then:*

- (i) *If $n(A) \leq d(A)$ and $n(A) < \infty$, then there exists an everywhere defined operator B in \mathfrak{H} with $\dim \operatorname{ran} B \leq n(A)$ such that the relation $A_1 = A - B$ satisfies $\ker A_1 = \{0\}$.*
- (ii) *If $d(A) \leq n(A)$ and $d(A) < \infty$, then there exists an everywhere defined operator B in \mathfrak{H} with $\dim \operatorname{ran} B \leq d(A)$ such that the relation $A_1 = A - B$ satisfies $\operatorname{ran} A_1 = \mathfrak{H}$.*
- (iii) *If $n(A) = d(A) < \infty$, then there exists an everywhere defined operator B in \mathfrak{H} with $\dim \operatorname{ran} B \leq n(A) = d(A)$ such that the relation $A_1 = A - B$ satisfies $n(A_1) = d(A_1) = 0$.*

Proof. (i) If $n(A) = 0$, then consider $B = 0$ and $A_1 = A$. Assume now that $1 \leq p = n(A)$ and let x_1, \dots, x_p be a basis of $\ker A$. Choose linear functionals x'_1, \dots, x'_p such that $x'_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, p$, and choose elements y_1, \dots, y_p in \mathfrak{H} such that the corresponding cosets $[y_1], \dots, [y_p] \in \mathfrak{H}/\operatorname{ran} A$ are linearly independent (such elements exist since $p \leq d(A)$). Define the operator B in \mathfrak{H} by

$$Bx = \sum_{i=1}^p x'_i(x) y_i, \quad x \in \mathfrak{H},$$

so that $\operatorname{dom} B = \mathfrak{H}$ and $\dim \operatorname{ran} B \leq n(A)$. Define the relation A_1 in \mathfrak{H} by $A_1 = A - B$, so that

$$A_1 = \{\{x, y - Bx\} : \{x, y\} \in A\},$$

To show that $\ker A_1 = \{0\}$, let $x \in \ker A_1$. Then $\{x, y\} \in A$ with $y = Bx \in \operatorname{ran} A$. Hence $y = \sum_{i=1}^p x'_i(x) y_i$ and $x'_i(x) = 0$, $1 \leq i \leq p$, which shows that $Bx = 0$. Therefore $x \in \ker A$ and then $x = \sum_{i=1}^p a_i x_i$ for some constants $a_i \in \mathbb{K}$. Since $0 = x'_i(x) = a_i$ it follows that $x = 0$. Therefore $\ker A_1 = \{0\}$.

(ii) If $d(A) = 0$, then consider $B = 0$ and $A_1 = A$. Assume now that $1 \leq q = d(A)$ and let x_1, \dots, x_q be a set of q linearly independent elements of $\ker A$ (such elements exist, because $q \leq n(A)$). Choose linear functionals x'_1, \dots, x'_q such that $x'_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, q$, and choose elements y_1, \dots, y_q in \mathfrak{H} such that the corresponding cosets $[y_1], \dots, [y_q] \in \mathfrak{H}/\operatorname{ran} A$ determine a basis of $\mathfrak{H}/\operatorname{ran} A$. Define the operator B in \mathfrak{H} by

$$Bx = \sum_{i=1}^q x'_i(x) y_i, \quad x \in \mathfrak{H},$$

so that $\operatorname{dom} B = \mathfrak{H}$ and $\dim \operatorname{ran} B \leq d(A)$. Define the relation A_1 in \mathfrak{H} by $A_1 = A - B$, so that

$$A_1 = \{\{x, y - Bx\} : \{x, y\} \in A\}.$$

To show that $\operatorname{ran} A_1 = \mathfrak{H}$ let $y \in \mathfrak{H}$. The subspace generated by y_1, \dots, y_q is a complement of $\operatorname{ran} A$ in \mathfrak{H} , so that y can be written as

$$y = \sum_{i=1}^q a_i y_i + g,$$

for some $g \in \text{ran } A$. Let $f \in \text{dom } A$ such that $\{f, g\} \in A$ and define $h \in \text{dom } A$ by

$$h = f - \sum_{i=1}^q [a_i + x'_i(f)]x_i.$$

Since $x_i \in \ker A$ it follows that $\{h, g\} \in A$, so that

$$\left\{ h, g + \sum_{i=1}^q a_i y_i \right\} \in A_1.$$

This shows that $\{h, y\} \in A_1$ and $y \in \text{ran } A_1$. Therefore $\text{ran } A_1 = \mathfrak{H}$.

(iii) If $n(A) = d(A)$ the constructions of B and A_1 in (i) and (ii) are identical, so that the assertion in (iii) follows. \square

Theorem 9.2. *Let A be a relation in a linear space \mathfrak{H} . Assume that there exist an everywhere defined operator B in \mathfrak{H} with $\dim \text{ran } B < \infty$ and a relation A_1 in \mathfrak{H} with $n(A_1) = d(A_1) = 0$ such that $A = A_1 + B$. Then the relation S defined by $S = I + A_1^{-1}B$ is (the graph of) an everywhere defined operator with $n(S) = n(A) = d(A) = d(S)$ and $\alpha(S) = \delta(S) < \infty$.*

Proof. With the relation A_1 and the operator B the formal product $A_1^{-1}B$ is given as the relation

$$A_1^{-1}B = \{\{x, y\} : \{y, Bx\} \in A_1\}, \quad (9.1)$$

since $\text{dom } B = \mathfrak{H}$. Observe that $\text{ran } A_1^{-1}B \subset \text{dom } A_1 = \text{dom } A$. Now $d(A_1) = 0$ means $\text{ran } A_1 = \mathfrak{H}$, so that $\text{dom } A_1^{-1}B = \mathfrak{H}$; while $n(A_1) = 0$ means that $\text{mul } A_1^{-1}B = \{0\}$. Hence $A_1^{-1}B$ is (the graph of) an everywhere defined operator. Note that A_1^{-1} itself is the graph of an everywhere defined operator, so that the formal product $A_1^{-1}B$ is the product of the everywhere defined operators A_1^{-1} and B . Clearly, also $S = I + A_1^{-1}B$ is (the graph of) an everywhere defined operator.

The operator S connects the relations A and A_1 as follows $A = A_1 S$. To see the inclusion $A_1 S \subset A$, let $\{x, y\} \in A_1 S$, so that $\{x, z\} \in S$ and $\{z, y\} \in A_1$ for some $z \in \mathfrak{H}$. Then $\{x, z\} \in I + A_1^{-1}B$, which implies that $\{x, z - x\} \in A_1^{-1}B$. Since $\{z - x, Bx\} \in A_1$ and $\{z, y\} \in A_1$, it follows that $\{x, y - Bx\} \in A_1$ and, as $\{x, Bx\} \in B$, this implies that

$$\{x, y\} = \{x, y - Bx + Bx\} \in A_1 + B = A.$$

Hence $A_1 S \subset A$. To see the converse inclusion, let $\{x, y\} \in A$ so that $\{x, y - Bx\} \in A_1$. Let z be the uniquely defined element $\{x, z\} \in A_1^{-1}B$. Then

$$\{x, x + z\} \in S \quad \text{and} \quad \{z, Bx\} \in A_1. \quad (9.2)$$

The last statement in (9.2) with $\{x, y - Bx\} \in A_1$ implies that $\{x + z, y\} \in A_1$, which together with the first result in (9.2) leads to $\{x, y\} \in A_1 S$. Hence $A \subset A_1 S$.

It follows from (9.1) that $\dim \text{ran } A_1^{-1}B \leq \dim \text{ran } B < \infty$. Therefore Theorem 7.4 implies that $n(S) = d(S) < \infty$ and that $\alpha(S) = \delta(S) < \infty$.

In order to show that $n(A) = n(S)$ it suffices to show that $\ker A = \ker S$. Let $x \in \ker A$. Then $\{x, 0\} \in A = A_1 S$, so that $\{x, y\} \in S$, $\{y, 0\} \in A_1$, for some $y \in \mathfrak{H}$. Since $\ker A_1 = \{0\}$ it follows that $y = 0$, so that $x \in \ker S$. Hence $\ker A \subset \ker S$. Conversely, if $x \in \ker S$, then $\{x, 0\} \in S$ and since $\{0, 0\} \in A_1$ it follows that $\{x, 0\} \in A_1 S = A$. Hence $\ker S \subset \ker A$. Therefore $\ker A = \ker S$, which shows that $n(A) = n(S)$.

It remains to show $d(A) = d(S)$. This will be proved by showing $d(A) \leq d(S)$ and $d(A) \geq d(S)$, respectively.

First it will be shown that $d(A) \leq d(S)$. The case $d(A) = 0$ is clear. Assume that $d(A) \geq 1$ and let \mathfrak{N} be a subspace, complementary to $\text{ran } A$ in \mathfrak{H} , i.e., $\mathfrak{H} = \text{ran } A \oplus \mathfrak{N}$. Let y_i , $1 \leq i \leq n$, be linearly independent elements of \mathfrak{N} . Since $\text{ran } A_1 = \mathfrak{H}$, it follows that $\{x_i, y_i\} \in A_1$ for some $x_i \in \text{dom } A_1 = \text{dom } A$. In order to show that $n \leq d(S)$ it suffices to show that the elements $[x_i]$, $1 \leq i \leq n$, of $\mathfrak{H}/\text{ran } S$ are linearly independent. Assume that $\sum_{i=1}^n a_i [x_i] = 0$, which means that $\sum_{i=1}^n a_i x_i \in \text{ran } S$, so that $\sum_{i=1}^n a_i x_i = Sv = v + A_1^{-1}Bv$ for some $v \in \mathfrak{H}$. Since $x_i \in \text{dom } A$, $1 \leq i \leq n$, and $A_1^{-1}Bv \in \text{dom } A$ it follows that $v \in \text{dom } A = \text{dom } A_1$, so that $\{v, u\} \in A_1$ for some $u \in \mathfrak{H}$. Furthermore, since $A_1^{-1}Bv \in \text{dom } A = \text{dom } A_1$ it follows that $\{A_1^{-1}Bv, w\} \in A_1$ for some $w \in \mathfrak{H}$. Observe that

$$\left\{0, \sum_{i=1}^n a_i y_i - u - w\right\} = \sum_{i=1}^n \{a_i x_i, a_i y_i\} - \{v, u\} - \{A_1^{-1}Bv, w\} \in A_1,$$

from which it follows that

$$\sum_{i=1}^n a_i y_i = u + w + m$$

for some $m \in \text{mul } A_1 = \text{mul } A \subset \text{ran } A$. Clearly, $\{v, u + Bv\} \in A_1 + B = A$, so that $u + Bv \in \text{ran } A$. It follows from $\{A_1^{-1}Bv, w\} \in A_1$ that $\{w, A_1^{-1}Bv\} \in A_1^{-1}$, or, equivalently, that $A_1^{-1}(w - Bv) = 0$, so that $w - Bv \in \ker A_1^{-1} = \text{mul } A_1 = \text{mul } A$. Therefore

$$\sum_{i=1}^n a_i y_i = (u + Bv) + (w - Bv) + m \in \text{ran } A.$$

Since $\sum_{i=1}^n a_i y_i \in \mathfrak{N}$ and $\mathfrak{N} \cap \text{ran } A = \{0\}$, it follows that $\sum_{i=1}^n a_i y_i = 0$. Hence $a_i = 0$, $1 \leq i \leq n$. Thus the elements $[x_i]$, $1 \leq i \leq n$, of $\mathfrak{H}/\text{ran } S$ are linearly independent. Therefore, $d(A) = \dim \mathfrak{N} \leq d(S)$.

Next it will be shown that $d(S) \leq d(A)$. Let $m = \delta(S)$. If $m = 0$, then $\text{ran } S = \mathfrak{H}$ and $d(S) = 0$. Hence, assume that $m \geq 1$. It follows from Lemma 5.6(ii) (applied to the everywhere defined operator S , $\delta(S) = m$, and $k = 1$) that there exists a subspace \mathfrak{M} of \mathfrak{H} such that $\mathfrak{M} \subset \ker S^m$, $\mathfrak{M} \cap \text{ran } S = \{0\}$, and $\mathfrak{H} = \text{ran } S \oplus \mathfrak{M}$. Let $q = d(S)$, let $\{x_i : 1 \leq i \leq q\}$ be a basis for \mathfrak{M} , and denote by T the operator $A_1^{-1}B$. Since

$$0 = S^m x_i = (I + T)^m x_i, \quad 1 \leq i \leq q,$$

it follows that $x_i \in \text{ran } T \subset \text{dom } A_1 = \text{dom } A$, so that $\{x_i, y_i\} \in A_1$ for some $y_i \in \mathfrak{H}$, $1 \leq i \leq q$. It will be shown that the elements $[y_i]$, $1 \leq i \leq q$, of $\mathfrak{H}/\text{ran } A$ are linearly independent. Assume that $\sum_{i=1}^q c_i [y_i] = 0$, so that $u' = \sum_{i=1}^q c_i y_i \in \text{ran } A$. Then $\{u, u'\} \in A$ for some $u \in \mathfrak{H}$ and thus $\{u, u' - Bu\} \in A_1$. Since also $\{\sum_{i=1}^q c_i x_i, u'\} \in A_1$ it follows that

$$\left\{\sum_{i=1}^q c_i x_i - u, Bu\right\} = \left\{\sum_{i=1}^q c_i x_i, u'\right\} - \{u, u' - Bu\} \in A_1,$$

which implies $\sum_{i=1}^q c_i x_i - u = A_1^{-1}Bu$. Therefore

$$\sum_{i=1}^q c_i x_i = (I + A_1^{-1}B)u = Su \in \text{ran } S.$$

Since $\sum_{i=1}^q c_i x_i \in \mathfrak{M}$ it follows that $\sum_{i=1}^q c_i x_i \in \text{ran } S \cap \mathfrak{M} = \{0\}$. Hence $\sum_{i=1}^q c_i x_i = 0$, so that $c_i = 0$, $1 \leq i \leq q$. Thus the elements $[y_i]$, $1 \leq i \leq q$, of $\mathfrak{H}/\text{ran } A$ are linearly independent. This implies that $q = d(S) \leq d(A)$. \square

Theorem 9.3. Let A be a relation in a linear space \mathfrak{H} . Assume that \mathfrak{M} and \mathfrak{N} are two complementary subspaces of \mathfrak{H} such that A is completely reduced by the pair $(\mathfrak{M}, \mathfrak{N})$. Furthermore, assume that $\mathfrak{N} \subset \text{dom } A$, $\dim \mathfrak{N} < \infty$, $A_{\mathfrak{N}}$ is an operator, and $A_{\mathfrak{M}}$ a relation with $n(A_{\mathfrak{M}}) = d(A_{\mathfrak{M}}) = 0$. Let P and Q be the projections of \mathfrak{H} onto \mathfrak{M} along \mathfrak{N} and onto \mathfrak{N} along \mathfrak{M} , respectively. Assume that $\lambda \in \mathbb{K} \setminus \{0\}$ and define the relations A_1 and B in \mathfrak{H} by

$$A_1 = \{\{x, Py - \lambda Qx\} : \{x, y\} \in A\}, \quad (9.3)$$

and

$$B = \{\{x, A_{\mathfrak{N}}Qx + \lambda Qx\} : x \in \mathfrak{H}\}. \quad (9.4)$$

Then B is an everywhere defined operator in \mathfrak{H} with $\text{ran } B \subset \mathfrak{N}$ and $\dim \text{ran } B < \infty$, the relation A_1 in \mathfrak{H} has the property that $n(A_1) = d(A_1) = 0$, and $A = A_1 + B$. Furthermore, $\text{mul } A = \text{mul } A_1$, $BA_1 \subset A_1B$, and the following two equalities:

$$A = \frac{1}{\lambda}(\lambda - B)A_1, \quad (9.5)$$

and

$$\text{ran}(\lambda - B) = \text{ran } A, \quad (9.6)$$

hold true.

Proof. The definition (9.4) shows that $B = (A_{\mathfrak{N}} + \lambda)Q$. Hence B is an everywhere defined linear operator in \mathfrak{H} . Moreover, $\text{ran } Q \subset \mathfrak{N}$ and $\text{ran } A_{\mathfrak{N}} \subset \mathfrak{N}$ imply that $\text{ran } B \subset \mathfrak{N}$. Then also $\dim \text{ran } B < \infty$.

Now it will be shown that $n(A_1) = 0$. Let $x \in \ker A_1$, there exists an element $y \in \mathfrak{H}$ such that $\{x, y\} \in A$ and $Py - \lambda Qx = 0$. Since $Py \in \mathfrak{M}$ and $\lambda Qx \in \mathfrak{N}$ it follows that $Py = Qx = 0$. Recall that $\{Px, Py\} \in A_{\mathfrak{M}}$ and $\{Qx, Qy\} \in A_{\mathfrak{N}}$. Therefore $Px \in \ker A_{\mathfrak{M}}$ and, since $n(A_{\mathfrak{M}}) = 0$, this implies that $Px = 0$. Therefore, $x = Px + Qx = 0$. Hence $\ker A_1 = \{0\}$, which shows that $n(A_1) = 0$.

Next it will be shown that $d(A_1) = 0$. Let $x \in \mathfrak{H}$, so that $x = x_1 + x_2$, for some $x_1 \in \mathfrak{M}$ and some $x_2 \in \mathfrak{N}$. Since $x_2 \in \mathfrak{N} \subset \text{dom } A$, it follows that $\{x_2, A_{\mathfrak{N}}x_2\} \in A_{\mathfrak{N}} \subset A$. Then, by (9.3), $\{x_2, -\lambda x_2\} \in A_1$, or, equivalently,

$$\left\{-\frac{1}{\lambda}x_2, x_2\right\} \in A_1. \quad (9.7)$$

Since $d(A_{\mathfrak{M}}) = 0$ and $x_1 \in \mathfrak{M}$, it follows that $\{y, x_1\} \in A_{\mathfrak{M}} \subset A$ for some $y \in \mathfrak{M}$, so that by (9.3)

$$\{y, x_1\} \in A_1. \quad (9.8)$$

Now, (9.7) and (9.8) lead to

$$\left\{y - \frac{1}{\lambda}x_2, x\right\} = \{y, x_1\} + \left\{-\frac{1}{\lambda}x_2, x_2\right\} \in A_1,$$

which shows that $x \in \text{ran } A_1$. Thus $\text{ran } A_1 = \mathfrak{H}$, so that $d(A_1) = 0$.

Furthermore, the equality $A = A_1 + B$ is a direct consequence of (9.3) and (9.4).

Now the identity $\text{mul } A = \text{mul } A_1$ will be shown. First note that $\text{mul } A \subset \mathfrak{M}$. To see this, let $y \in \text{mul } A$ so that $\{0, y\} \in A$ for some $y \in \mathfrak{M}$. Then since $A = A_{\mathfrak{M}} \oplus A_{\mathfrak{N}}$ one has $\{0, y\} = \{u, u'\} \dot{+} \{v, v'\}$ with $\{u, u'\} \in A_{\mathfrak{M}}$ and $\{v, v'\} \in A_{\mathfrak{N}}$. Now $u + v = 0$ implies $u = 0$ and $v = 0$. Since $A_{\mathfrak{N}}$ is an operator it follows that $v' = 0$. Hence $\{0, y\} \in A_{\mathfrak{M}}$ and, in particular, $y \in \mathfrak{M}$. Hence

$\text{mul } A \subset \mathfrak{M}$. In particular, this means for $\{0, y\} \in A$ that $Py = y$ which shows that $\{0, y\} \in A_1$. Hence $\text{mul } A \subset \text{mul } A_1$. To show the converse inclusion, let $y \in \text{mul } A_1$ so that $y = Py'$ for some $y' \in \text{mul } A \subset \mathfrak{M}$. Thus $y = y'$ and hence $\text{mul } A_1 \subset \text{mul } A$. Therefore the identity $\text{mul } A = \text{mul } A_1$ has been shown.

In order to prove the inclusion $BA_1 \subset A_1B$, let $\{x, y\} \in BA_1$. Then $\{x, z\} \in A_1$ and $\{z, y\} \in B$ for some $z \in \mathfrak{H}$. Then $z = Px' - \lambda Qx$ for some $\{x, x'\} \in A$, which implies that

$$y = Bz = A_{\mathfrak{N}}Qz + \lambda Qz = -\lambda A_{\mathfrak{N}}Qx - \lambda^2 Qx. \quad (9.9)$$

Furthermore, $w = A_{\mathfrak{N}}Qx + \lambda Qx \in \mathfrak{N} \subset \text{dom } A$, so that $\{w, w'\} \in A$ for some $w' \in \mathfrak{H}$. Then

$$\{w, w'\} = \{w, A_{\mathfrak{N}}w\} + \{0, Pw'\},$$

where $A_{\mathfrak{N}}w = Qw'$ and $Pw' \in \text{mul } A = \text{mul } A_1$. Since $\{w, w'\} \in A$, it follows that $\{w, Pw' - \lambda Qw\} \in A_1$ and since $Pw' \in \text{mul } A_1$ it follows that $\{w, -\lambda Qw\} \in A_1$, which leads to

$$\{A_{\mathfrak{N}}Qx + \lambda Qx, -\lambda A_{\mathfrak{N}}Qx - \lambda^2 Qx\} \in A_1. \quad (9.10)$$

It follows from (9.4) that

$$\{x, A_{\mathfrak{N}}Qx + \lambda Qx\} \in B. \quad (9.11)$$

Clearly, (9.9)–(9.11) leads to $\{x, y\} \in A_1B$. Therefore $BA_1 \subset A_1B$.

Next it is shown that (9.5) holds true. First the inclusion

$$A \subset \frac{1}{\lambda}(\lambda - B)A_1 \quad (9.12)$$

will be proved. Let $\{x, y\} \in A$, so that

$$\{x, Py - \lambda Qx\} \in A_1. \quad (9.13)$$

Since $Py - \lambda Qx \in \text{dom } B = \mathfrak{H}$ it follows that

$$\{Py - \lambda Qx, -\lambda A_{\mathfrak{N}}Qx - \lambda^2 Qx\} \in B.$$

Now recall $\{x, y\} = \{Px, Py\} \hat{+} \{Qx, Qy\}$ so that $y = Py + A_{\mathfrak{N}}Qx$. Therefore

$$\{Py - \lambda Qx, -\lambda y\} = \{Py - \lambda Qx, -\lambda A_{\mathfrak{N}}Qx - \lambda Py\} \in B - \lambda,$$

which shows that

$$\{Py - \lambda Qx, y\} \in \frac{1}{\lambda}(\lambda - B). \quad (9.14)$$

It follows from (9.13) and (9.14) that $\{x, y\} \in \frac{1}{\lambda}(\lambda - B)A_1$. Hence the inclusion (9.12) has been proved. In order to prove the converse inclusion

$$\frac{1}{\lambda}(\lambda - B)A_1 \subset A, \quad (9.15)$$

let $\{x, y\} \in \frac{1}{\lambda}(\lambda - B)A_1$, so that $\{x, z\} \in A_1$ and $\{z, y\} \in \frac{1}{\lambda}(\lambda - B)$ for some $z \in \mathfrak{H}$. Then $Bz = \lambda(z - y)$ and by (9.4) it follows that

$$y = -\frac{1}{\lambda}A_{\mathfrak{N}}Qz + Pz. \quad (9.16)$$

Since $\{x, z\} \in A_1$, it follows that

$$z = Py' - \lambda Qx \quad \text{for some } \{x, y'\} \in A. \quad (9.17)$$

Substitute (9.17) into (9.16) to obtain

$$y = A_{\mathfrak{N}} Qx + Py' = Qy' + Py' = y',$$

where the equality $A_{\mathfrak{N}} Qx = Qy'$ has been used. Therefore $\{x, y\} = \{x, y'\} \in A$. Hence (9.15) has been proved. It follows from (9.12) and (9.15) that (9.5) holds true.

In order to prove (9.6), observe that the equality (9.5) implies that

$$\text{ran } A \subset \text{ran}(\lambda - B).$$

Hence it suffices to show that

$$\text{ran}(\lambda - B) \subset \text{ran } A.$$

Therefore, let $y \in \text{ran}(\lambda - B)$. Then $y = (\lambda - B)x$ for some $x \in \mathfrak{H}$, so that $\{x, \lambda x - y\} \in B$, which leads to

$$y = \lambda Px - A_{\mathfrak{N}} Qx. \quad (9.18)$$

Clearly,

$$\{Qx, A_{\mathfrak{N}} Qx\} \in A_{\mathfrak{N}} \subset A. \quad (9.19)$$

Since $\lambda Px \in \mathfrak{M}$ and $d(A_{\mathfrak{M}}) = 0$, it follows that

$$\{z, \lambda Px\} \in A_{\mathfrak{M}} \subset A, \quad (9.20)$$

for some $z \in \mathfrak{H}$. A combination of (9.18)–(9.20) leads to

$$\{z - Qx, y\} = \{z, \lambda Px\} - \{Qx, A_{\mathfrak{N}} Qx\} \in A,$$

so that $y \in \text{ran } A$. Hence the inclusion $\text{ran}(\lambda - B) \subset \text{ran } A$ holds true. \square

Theorem 9.4. *Let A be a relation in a linear space \mathfrak{H} . Assume that there exist an everywhere defined operator B in \mathfrak{H} with $\dim \text{ran } B < \infty$ and a relation A_1 in \mathfrak{H} with $n(A_1) = d(A_1) = 0$ such that $A = A_1 + B$ and $BA_1 \subset A_1 B$. Then $n(A) = d(A) < \infty$ and $\alpha(A) = \delta(A) < \infty$.*

Proof. Clearly, Theorem 9.2 leads to $n(A) = d(A) < \infty$.

Since $n(A_1) = d(A_1) = 0$, it follows that A_1^{-1} is an everywhere defined operator in \mathfrak{H} . The operators B and A_1^{-1} commute. Indeed, let $x \in \mathfrak{H}$, so that $\{y, x\} \in A_1$ for some $y \in \text{dom } A_1 = \text{dom } A$ (due to $d(A_1) = 0$). Since $\{x, Bx\} \in B$ it follows that $\{y, Bx\} \in BA_1 \subset A_1 B$, so that $\{y, z\} \in B$ and $\{z, Bx\} \in A_1$ for some $z \in \mathfrak{H}$. Therefore $z = By$ and $\{Bx, By\} \in A_1^{-1}$, so that $By = A_1^{-1} Bx$. Since $\{y, x\} \in A_1$ it follows that $y = A_1^{-1} x$, and then $BA_1^{-1} x = A_1^{-1} Bx$, which shows that $BA_1^{-1} = A_1^{-1} B$.

Next it is shown that

$$\ker A^k \subset \text{ran } B \quad \text{for all } k \in \mathbb{N}. \quad (9.21)$$

This inclusion holds for $k = 0$. Now assume that $\ker A^k \subset \text{ran } B$ for certain $k \in \mathbb{N}$. Let $x \in \ker A^{k+1}$ so that $\{x, y\} \in A$ and $\{y, 0\} \in A^k$. Thus $y \in \ker A^k \subset \text{ran } B$, which implies that $y = Bz$ for some $z \in \mathfrak{H}$. It follows from $\{x, y\} \in A = A_1 + B$ that $\{x, B(z - x)\} = \{x, y - Bx\} \in A_1$. Therefore, $\{B(z - x), x\} \in A_1^{-1}$, so that, since B and A_1^{-1} commute,

$$x = A_1^{-1} B(z - x) = BA_1^{-1}(z - x) \in \text{ran } B,$$

which leads to $\ker A^{k+1} \subset \text{ran } B$. Hence (9.21) has been shown.

The inclusion in (9.21) shows that $n(A^k) \leq \dim \operatorname{ran} B < \infty$ for all $k \in \mathbb{N}$. Then, by Lemma 6.1 it follows that $\alpha(A) < \infty$ and hence by Theorem 6.11 it follows that $\alpha(A) = \delta(A) < \infty$. \square

These decomposition results lead also to a completely reduced decomposition of a relation in a linear space.

Corollary 9.5. *Let A be a relation in a linear space \mathfrak{H} . Assume that \mathfrak{M} and \mathfrak{N} are complementary subspaces of \mathfrak{H} such that A is completely reduced by the pair $(\mathfrak{M}, \mathfrak{N})$. Furthermore, assume that $\mathfrak{N} \subset \operatorname{dom} A$, $\dim \mathfrak{N} < \infty$, $A_{\mathfrak{N}}$ is an operator, and $A_{\mathfrak{M}}$ is a relation with $n(A_{\mathfrak{M}}) = d(A_{\mathfrak{M}}) = 0$. Then $q = \delta(A) < \infty$ and A is completely reduced by the pair $(\operatorname{ran} A^q, \ker A^q)$.*

Proof. It follows from Theorems 9.3 and 9.4 that $n(A) = d(A) < \infty$ and $\alpha(A) = \delta(A) < \infty$.

Next it is shown that $\mathfrak{R}_c(A) = \{0\}$. Assume that

$$\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_{s-1}, x_s\}, \{x_s, 0\} \quad (9.22)$$

is a nontrivial singular chain in A . By assumption one has the direct sum decomposition

$$\{0, x_1\} = \{e_1, A_{\mathfrak{N}}e_1\} \hat{+} \{f_1, g_1\}, \quad e_1 \in \mathfrak{N}, \quad \{f_1, g_1\} \in A_{\mathfrak{M}}.$$

This implies that $e_1 = f_1 = 0$ so that $x_1 = g_1 \in \mathfrak{M}$. Similarly, the next element in the chain has a decomposition

$$\{x_1, x_2\} = \{e_2, A_{\mathfrak{N}}e_2\} \hat{+} \{f_2, g_2\}, \quad e_2 \in \mathfrak{N}, \quad \{f_2, g_2\} \in A_{\mathfrak{M}}.$$

Now $x_1 \in \mathfrak{M}$ implies $e_2 = 0$ so that $x_2 = g_2 \in \mathfrak{M}$. By induction one concludes $x_1, \dots, x_s \in \mathfrak{M}$. Therefore the chain in (9.22) is a singular chain for $A_{\mathfrak{M}}$. However the condition $n(A_{\mathfrak{M}}) = 0$ implies that $x_i = 0$, $1 \leq i \leq s$, cf. Lemma 5.3. This contradicts the assumption that the chain is nontrivial. Hence $\mathfrak{R}_c(A) = \{0\}$.

The conclusion follows now by Theorem 8.3(ii). \square

10. Examples

10.1. Singular chains

In a number of results in the present paper it was assumed that $\mathfrak{R}_c(A) = \{0\}$, in other words, it was assumed that the relation A does not have nontrivial singular chains. It is now shown that without this condition those results are not valid anymore.

Example 10.1. Let $\mathfrak{H} = \operatorname{span}\{e_1\}$ with $e_1 \neq 0$ and define the relation A in \mathfrak{H} by

$$A = \operatorname{span}\{\{0, e_1\}, \{e_1, 0\}\}.$$

The nullity and defect of A are given by

$$n(A) = 1, \quad d(A) = 0.$$

Moreover $A^2 = A = \mathfrak{H} \times \mathfrak{H}$ and the ascent and descent of A are given by

$$\alpha(A) = 1, \quad \delta(A) = 0.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Note that

$$\ker A \cap \operatorname{ran} A = \mathfrak{H}, \quad \operatorname{mul} A \cap \operatorname{dom} A = \mathfrak{H}.$$

Hence the conclusions of Lemma 4.4 (with $i = k = 1$), Lemma 5.5(ii), Theorems 5.7, 5.8(ii), 6.4(i) (with $\mathfrak{M} = \mathfrak{H}$), Theorem 6.5, Corollary 6.8, Theorems 6.9 and 7.4 (as $A - \lambda = A$) fail in the presence of nontrivial singular chains.

Example 10.2. Let $\mathfrak{H} = \operatorname{span}\{e_1, e_2\}$ with e_1, e_2 linearly independent and define the relation A in \mathfrak{H} by

$$A = \operatorname{span}\{\{0, e_1\}, \{e_1, 0\}\}.$$

The nullity and defect of A are given by

$$n(A) = 1, \quad d(A) = 1.$$

Moreover $A^2 = A = \mathfrak{H} \times \mathfrak{H}$ and the ascent and descent of A are given by

$$\alpha(A) = 1, \quad \delta(A) = 1.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Note that

$$\operatorname{ran} A + \operatorname{dom} A = \operatorname{span}\{e_1\} \neq \mathfrak{H}.$$

Hence the conclusions of Theorem 6.13 and Theorem 8.3(ii) fail in the presence of nontrivial singular chains.

Example 10.3. Let $\mathfrak{H} = \operatorname{span}\{e_1, e_2\}$ with e_1, e_2 linearly independent and define the relation A by

$$A = \operatorname{span}\{\{0, e_1\}, \{e_1, 0\}, \{e_2, 0\}\}.$$

The nullity and defect of A are given by

$$n(A) = 2, \quad d(A) = 1,$$

Moreover $A^2 = A$ and the ascent and descent of A are given by

$$\alpha(A) = 1, \quad \delta(A) = 1.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Note that

$$\operatorname{ran} A \cap \ker A = \operatorname{span}\{e_1\}.$$

Hence the conclusion of Theorem 8.3(iii) fails in the presence of nontrivial singular chains.

Example 10.4. Let $\mathfrak{H} = \operatorname{span}\{e_1, e_2, e_3\}$ with e_1, e_2, e_3 linearly independent and define the relation A by

$$A = \operatorname{span}\{\{0, e_1\}, \{e_1, e_2\}, \{e_2, 0\}\}.$$

The nullity and defect of A are given by

$$n(A) = 1, \quad d(A) = 1.$$

Moreover

$$A^2 = A^3 = \operatorname{span}\{\{0, e_1\}, \{e_1, 0\}, \{0, e_2\}, \{e_2, 0\}\},$$

and the ascent and descent of A are given by

$$\alpha(A) = 2, \quad \delta(A) = 1.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Hence the conclusions of Theorem 6.11 and Lemma 7.3(iii) (as $A - \lambda = A$) fail in the presence of nontrivial singular chains.

Example 10.5. Let $\mathfrak{H} = \text{span}\{e_1, e_2, e_3\}$ with e_1, e_2, e_3 linearly independent, and let $\mathfrak{H}_1 = \text{span}\{e_1\}$ and $\mathfrak{H}_2 = \text{span}\{e_2, e_3\}$. Define the relations A_1 in \mathfrak{H}_1 and A_2 in \mathfrak{H}_2 by

$$A_1 = \text{span}\{\{0, e_1\}, \{e_1, 0\}\}, \quad A_2 = \text{span}\{\{e_2, 0\}\}.$$

As in Example 10.1 one has

$$n(A_1) = 1, \quad d(A_1) = 0, \quad \alpha(A_1) = 1, \quad \delta(A_1) = 0,$$

and $\mathfrak{R}_c(A_1) \neq \{0\}$. It is straightforward to see that

$$n(A_2) = 1, \quad d(A_2) = 2, \quad \alpha(A_2) = 1, \quad \delta(A_2) = 1.$$

Define the relation A in \mathfrak{H} by

$$A = \text{span}\{\{0, e_1\}, \{e_1, 0\}, \{e_2, 0\}\},$$

so that $A = A_1 \oplus A_2$. Observe that $A^2 = A$ and that the nullity, defect, ascent, and descent of A are given by

$$n(A) = 2, \quad d(A) = 2, \quad \alpha(A) = 1, \quad \delta(A) = 1.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Hence the conclusion of Theorem 8.1 (vi) fails in the presence of nontrivial singular chains.

Example 10.6. Let $\mathfrak{H} = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ with e_1, e_2, e_3, e_4, e_5 linearly independent and let $\mathfrak{H}_1 = \text{span}\{e_1, e_2\}$ and $\mathfrak{H}_2 = \text{span}\{e_3, e_4, e_5\}$. Define the relations A_1 in \mathfrak{H}_1 and A_2 in \mathfrak{H}_2 by

$$A_1 = \text{span}\{\{0, e_1\}, \{e_1, 0\}\}, \quad A_2 = \text{span}\{\{0, e_3\}, \{e_3, e_4\}, \{e_4, 0\}\}.$$

As in Example 10.1 one has

$$n(A_1) = 1, \quad d(A_1) = 1, \quad \alpha(A_1) = 1, \quad \delta(A_1) = 1,$$

and $\mathfrak{R}_c(A_1) \neq \{0\}$. As in Example 10.4 one has

$$n(A_2) = 1, \quad d(A_2) = 1, \quad \alpha(A_2) = 2, \quad \delta(A_2) = 1.$$

Define the relation A in \mathfrak{H} by

$$A = \text{span}\{\{0, e_1\}, \{e_1, 0\}, \{0, e_3\}, \{e_3, e_4\}, \{e_4, 0\}\},$$

so that $A = A_1 \oplus A_2$. Observe that

$$A^2 = A^3 = \text{span}\{\{0, e_1\}, \{e_1, 0\}, \{0, e_3\}, \{e_3, 0\}, \{0, e_4\}, \{e_4, 0\}\},$$

and that the nullity, defect, ascent, and descent of A are given by

$$n(A) = 2, \quad d(A) = 2, \quad \alpha(A) = 2, \quad \delta(A) = 1.$$

Clearly $\mathfrak{R}_c(A) \neq \{0\}$. Hence the conclusion of Theorem 8.1 (vii) fails in the presence of nontrivial singular chains.

10.2. Pairs of relations

Assume that A and B are relations in a linear space \mathfrak{H} such that $A \subset B$. When $\mathfrak{R}_c(B) = \{0\}$ it has been proved that $\alpha(A) \leq \alpha(B)$, cf. Lemma 5.9. Now it is shown that this inequality fails if $\mathfrak{R}_c(B) \neq \{0\}$. Furthermore the remarks following Lemma 5.9 are illustrated by means of examples.

Example 10.7. Let $\mathfrak{H} = \text{span}\{e_1, e_2\}$ with e_1, e_2 linearly independent and define the relations A and B by

$$A = \text{span}\{\{e_1, e_2\}, \{e_2, 0\}\}, \quad B = \text{span}\{\{e_1, 0\}, \{0, e_2\}, \{e_2, 0\}\},$$

so that $A \subset B$. Moreover,

$$A^2 = A^3 = \text{span}\{\{e_1, 0\}, \{e_2, 0\}\}, \quad B^2 = B,$$

and

$$\alpha(A) = 2, \quad \alpha(B) = 1.$$

Clearly $\mathfrak{R}_c(B) \neq \{0\}$. Therefore the conclusion of Lemma 5.9 fails in the presence of nontrivial singular chains.

Example 10.8. Let $\mathfrak{H}_1 = \text{span}\{e_1, e_2\}$ with e_1, e_2 linearly independent and define the relations A_1 and B_1 in \mathfrak{H}_1 by

$$A_1 = \text{span}\{\{e_1, e_2\}\}, \quad B_1 = \text{span}\{\{e_1, e_2\}, \{e_2, e_1\}\},$$

so that $A_1 \subset B_1$. Then $A_1^2 = \{0, 0\}$, $B_1^2 = I$, and $\mathfrak{R}_c(B_1) = \{0\}$. It is clear that $\delta(A_1) = 2$ and $\delta(B_1) = 0$.

Now let $\mathfrak{H}_2 = \text{span}\{e_1, e_2, e_3\}$ with e_1, e_2, e_3 linearly independent and define the relations A_2 and B_2 in \mathfrak{H}_2 by

$$A_2 = \text{span}\{\{0, e_1\}\}, \quad B_2 = \text{span}\{\{0, e_1\}, \{e_2, e_3\}\},$$

so that $A_2 \subset B_2$. Then $A_2^2 = A_2$, $B_2^2 = B_2^3 = A_2$, and $\mathfrak{R}_c(B_2) = \{0\}$. It is clear that $\delta(A_2) = 1$ and $\delta(B_2) = 2$.

Hence if relations A and B satisfy $A \subset B$ then although $\alpha(A) \leq \alpha(B)$ there is no analogous relation between the descents $\delta(A)$ and $\delta(B)$, even if $\mathfrak{R}_c(B)$ is trivial.

Example 10.9. Let $\mathfrak{H} = \text{span}\{e_1, e_2\}$ with e_1, e_2 linearly independent and define the relations A and B by

$$A = \text{span}\{\{e_2, e_1\}, \{e_1, 0\}\}, \quad B = \text{span}\{\{e_1, 0\}, \{e_2, 0\}\}.$$

Then for all $n \geq 2$

$$A^n = B^n = \text{span}\{\{e_1, 0\}, \{e_2, 0\}\}.$$

In particular, $\alpha(A) = 2$ and $\alpha(B) = 1$. However $\ker A^n \subset \ker B^n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence the condition $A \subset B$ in Lemma 5.9 cannot be replaced by the condition $\ker A^n \subset \ker B^n$ for all $n \in \mathbb{N} \cup \{0\}$, even if $\mathfrak{R}_c(B)$ is trivial.

References

- [1] R. Arens, Operational calculus of linear relations, *Pacific J. Math.* 11 (1961) 9–23.
- [2] A.G. Baskakov, K.I. Chernyshov, Spectral analysis of linear relations and degenerate operator semigroups, *Mat. Sb.* 193 (2003) 1573–1610.
- [3] S.R. Caradus, Operators of Riesz type, *Pacific J. Math.* 18 (1966) 61–71.
- [4] S.R. Caradus, Operators with finite ascent and descent, *Pacific J. Math.* 18 (1966) 437–449.
- [5] S.R. Caradus, Perturbation theory for generalized Fredholm operators, *Pacific J. Math.* 52 (1974) 11–15.
- [6] E.A. Coddington, Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.* 134 (1973).
- [7] R.W. Cross, *Multivalued Linear Operators*, Marcel Dekker, New York, 1998.
- [8] V.A. Derkach, S. Hassi, M.M. Malamud, H.S.V. de Snoo, Boundary relations and their Weyl families, *Trans. Amer. Math. Soc.* 358 (2006) 5351–5400.
- [9] A. Favini, A. Yagi, *Degenerate Differential Equations in Banach Spaces*, Marcel Dekker, New York–Basel–Hong Kong, 1999.
- [10] I.C. Gohberg, M.G. Kreĭn, The basic propositions on defect numbers, root numbers, and indices of linear operators, *Uspehi Mat. Nauk (N.S.)* 12 (no. 2 (74)) (1957) 43–118 (Russian) [English translation: *Amer. Math. Soc. Transl., Series 2*, 13 (1960) 185–264].
- [11] H. Heuser, Über Operatoren mit endlichen Defekten, Inaugural-Dissertation, Universität Tübingen, 1956.
- [12] M.A. Kaashoek, Ascent, descent, nullity and defect: a note on a paper by A.E. Taylor, *Math. Ann.* 72 (1967) 105–115.
- [13] M.A. Kaashoek, D.C. Lay, Ascent, descent, and commuting perturbations, *Trans. Amer. Math. Soc.* 169 (1972) 35–47.
- [14] T. Kato, Perturbation theory for nullity, deficiency, and other quantities of linear operators, *J. d'Analyse Math.* 11 (1958) 261–322.
- [15] J.-Ph. Labrousse, Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm, *Rend. Circ. Mat. Palermo* (2) 29 (2) (1980) 161–258.
- [16] J.-Ph. Labrousse, A. Sandovici, H.S.V. de Snoo, H. Winkler, The Kato decomposition of quasi-Fredholm relations, in preparation.
- [17] D.C. Lay, Spectral analysis using ascent, descent, nullity and defect, *Math. Ann.* 184 (1970) 197–214.
- [18] F. Riesz, Über lineare Functionalgleichungen, *Acta Math.* 41 (1918) 71–98.
- [19] A. Sandovici, H.S.V. de Snoo, H. Winkler, The structure of linear relations in Euclidean spaces, *Linear Algebra Appl.* 397 (2005) 141–169.
- [20] A.E. Taylor, *Introduction to Functional Analysis*, John Wiley & Sons, New York, 1958.
- [21] A.E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, *Math. Ann.* 163 (1966) 18–49.
- [22] A.E. Taylor, D.C. Lay, *Introduction to Functional Analysis*, second ed., John Wiley & Sons, New York, 1980.